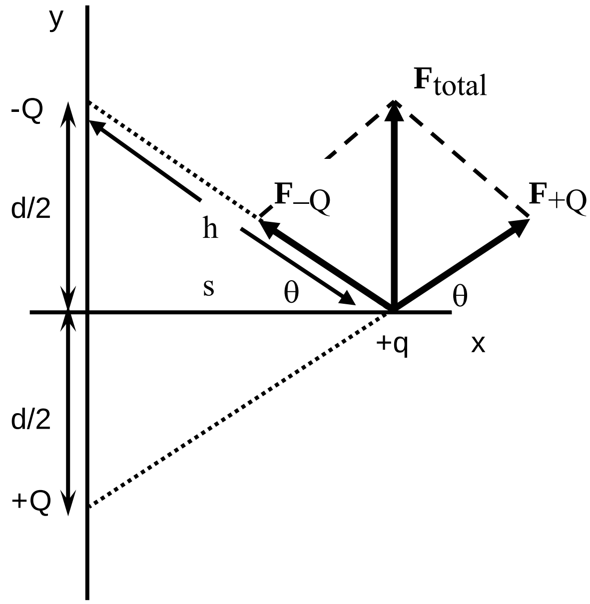
Question 1 (Marks)

(a)



The distance s is given by

$$s = \sqrt{h^2 + (a/2)^2}$$

Coulomb's law for charges q_1 and q_2 is

$$F = k_0 \frac{q_1 q_2}{r^2}$$

which gives the magnitude of the force on +q due to +Q and –Q

$$|\mathbf{F}_{+Q}| = |\mathbf{F}_{-Q}| = k_0 \frac{qQ}{h^2} = k_0 \frac{qQ}{\left[s^2 + (a/2)^2\right]}$$

and the magnitude of the total force is

$$|\mathbf{F}_{total}| = 2k_0 \frac{qQ}{h^2} \sin\theta = 2k_0 \frac{qQ}{\left[s^2 + (a/2)^2\right]} \sin\theta$$

substituting $\sin\theta = \dfrac{a/2}{h} = \dfrac{a/2}{\sqrt{s^2 + (a/2)^2}}$

$$|\mathbf{F}_{total}| = 2k_0 \frac{qQ}{\left[s^2 + (a/2)^2\right]} \frac{a/2}{\sqrt{s^2 + (a/2)^2}}$$

$$|\mathbf{F}_{total}| = k_0 \frac{aqQ}{\left[s^2 + (a/2)^2\right]^{3/2}}$$

By symmetry see that $\mathbf{F}_{total}$ is parallel to the y-axis, in $\hat{\mathbf{j}}$ direction so that

$$|\mathbf{F}_{total}| = k_0 \frac{aqQ}{\left[s^2 + (a/2)^2\right]^{3/2}} \hat{\mathbf{j}}$$

(b) The magnitude of the force between two charges q_1, q_2 is given by Coulomb's law:

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \qquad (1)$$

where ε_0 is the permittivity of free space. The force can be seen to arise from the electric field around charge q_i

$$E = \frac{F}{q} = \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r^2} \qquad (2)$$

If the charges are immersed in a medium other than free space, ε_0 is replaced in expressions (1) and (2) by the permittivity value ε of that medium. Thus, permittivity is a scaling factor for electric field strength in a medium and quantifies the electrostatic behaviour of the medium.

(c) Dielectric materials with high permittivity values are used in the production of capacitors. For example, in a parallel plate capacitor, the electric field between the plates carrying surface charge density σ is

$$E = \frac{\sigma}{\varepsilon}$$

a dielectric material with permittivity $\varepsilon >> 1$ inserted between the plates lowers the electric field between the plates, thus allowing greater charge to be stored at a given potential difference, i.e. a greater value of C (farads) is obtained for a given geometry.

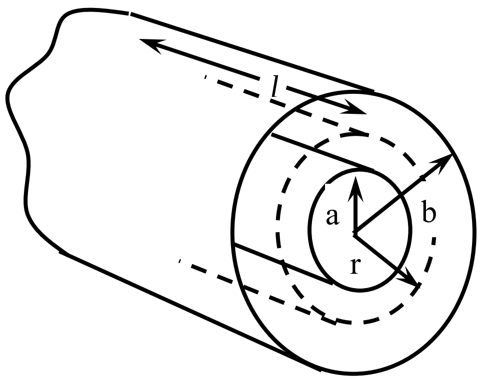
Other possible engineering examples: dielectric in a coaxial cable; the SiO_2 layer in CMOS microelectronics etc.

Question 2 (Marks)

(a) The cylinders have charge per unit length λ of opposite sign. Gauss' law is

$$\int_{\text{surface}} \mathbf{E}.d\mathbf{A} = \frac{1}{\varepsilon_0}\sum q_\bullet = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

We choose a cylindrical Gaussian surface radius r, length *l* – this suits the symmetry of the problem:



E is radial everywhere (in the direction of r) and the Gaussian surface has area

$$\int_{\text{surface}} dA = 2\pi r l$$

So Gauss' law is

$$\int_{\text{surface}} \mathbf{E}.d\mathbf{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0} \qquad \Rightarrow \qquad E(2\pi r l) = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

<u>For r < a,</u>

$$q_{\text{enclosed}} = 0 \qquad \Rightarrow \qquad E = 0$$

<u>For r > b,</u>

$$\left.\begin{array}{l}\text{inner conductor has linear charge density } +\lambda l \\ \text{outer conductor has linear charge density } -\lambda l\end{array}\right\}\ \text{net charge is zero}$$

$$q_{\text{enclosed}} = 0 \quad \Rightarrow \quad E = 0$$

<u>For a < r < b</u> Gauss' law gives

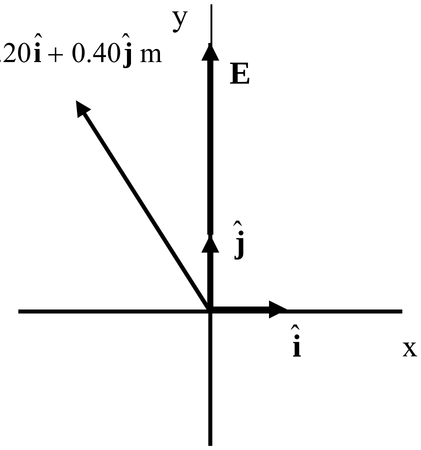
$$E(2\pi r l) = \frac{\lambda l}{\varepsilon_0}$$

so that

$$E = \frac{\lambda}{2\pi r\varepsilon_0}$$

Question 3 (Marks)

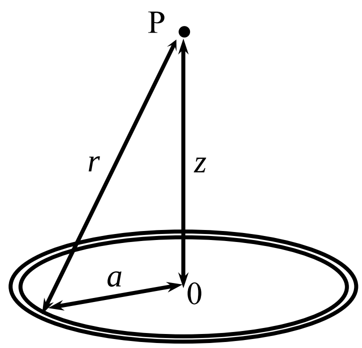
(a) A diagram is helpful (but does not carry any marks):



(i) $\Delta V = -\int_{path} \underset{\sim}{E} . d\underset{\sim}{s} = -\underset{\sim}{E} . \int d\underset{\sim}{s} = -\underset{\sim}{E} . \Delta\underset{\sim}{s} = -E_x \Delta s_x - E_y \Delta s_y = -40V$

(ii) Work done is $W = -q\Delta V = -e\Delta V = (-1.6x10^{-19} C)(-40V) = 6.4x10^{-18} J (= 40eV)$

The diagram shows a circular ring of uniform electric charge of radius *a*. The total charge on the ring is Q coulombs. Derive an expression in terms of *z* and *a* for the electric potential at a point P vertically above the centre of the ring, 0, as shown.



The potential at point P with position vector **r** is

$$V = k_0 \int \frac{dq}{r}$$

where $k_0 = \frac{1}{4\pi\varepsilon_0}$. We note from the geometry that

$$r = \sqrt{a^2 + z^2}$$

and then

$$V = k_0 \int \frac{dq}{r} = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{\sqrt{a^2 + z^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{a^2 + x^2}}$$

Question 4 (Marks)

(a) (i) The drift velocity comes from $I = nAve \quad \Rightarrow \quad v = \dfrac{I}{nAe}$

The carrier concentration is $n = \dfrac{\rho N_A}{M} = \dfrac{(8.9x10^3)(6.02x10^{23})}{63.5x10^{-3}} = 8.43x10^{28}\ m^{-3}$

$$v = \frac{I}{nAe} = \frac{100x10^{-3}}{(8.43x10^{28})(\pi\left(\frac{(0.2x10^{-3})^2}{4}\right)1.6x10^{-19}} = 2.36x10^{-4} ms^{-1} \approx 0.2\,mm.s^{-1}$$

(ii) This is a surprisingly small speed when compared to magnitude of other electronic speeds and a tiny fraction of the speed of light.

(b) (i) The circumference of the coil former is $2\pi r = 2\pi(5x10^{-3}) = 0.0314\ m$.

The 2m length of wire will form a coil 5mm in diameter with $\dfrac{2.0}{0.0314} = 63.69 \approx 64$ turns in a length $(64)(0.2x10^{-3})\ m = 0.0128\ m$.

The defining equation for inductance (air core) is

$$N\Phi = Li$$

so

$$L = \frac{N\Phi}{i} = \frac{NBA}{I} \approx \frac{\mu_0 N^2 A}{l}$$

and with iron former,

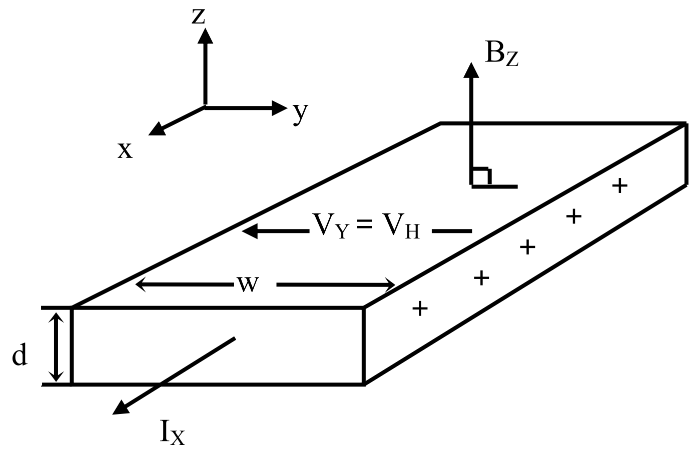
$$L = \frac{\mu_0 \mu_r N^2 A}{l}$$

$$L \approx \frac{(4\pi x10^{-7})(600)(64)^2(\pi(5x10^{-3})^2/4)}{0.0128} = 1.49x10^{-2}\ H = 14.9\ mH$$

(ii) Without iron former L is just reduced by factor $1/\mu_r = 1/600$.

$$L_{air} = \frac{1.49x10^{-2}}{600}\ H = 2.48x10^{-5}\ H = 24.8\ \mu H$$

(c) For a Hall effect measurement, the arrangement is:



Conventional current I in +x direction, electrons move toward –x direction, this is direction of v. Force on charge carriers is $\mathbf{F}_m$,

$$\mathbf{F}_m = q(\mathbf{v}x\mathbf{B})$$

For electrons, $q = -e$; $\mathbf{F}_m$ pushes electrons in –y direction leaving net +ve charge on one of the long faces, as shown. This charge separation produces the Hall electric field and a Hall voltage, V_H in direction shown.

When the Hall voltage is established the magnetic force on the electrons is balanced by the electric force due to the Hall field so that

$$eE_y = -B_z e v_x$$

From $I = nAve$ we see the current density J is

$$J_x = \frac{I_x}{A} = nev_x$$

The subscripts on J, I and v give their directions. Since $J = \sigma E$ where σ is the conductivity, we have

$$J_x = \frac{I}{A} = \sigma E_x = nev_x$$

Rearranging these we find that

$$E_y = -\frac{B_z J_x}{ne} = -\frac{B_z E_x \sigma}{ne}$$

Note that

$$E_y = \frac{V_y}{w} = \frac{V_H}{w}$$

where w is the width of the specimen in metres.

So, if the current, I, and magnetic field, B, are known, measurement of the Hall voltage V_H gives us the electron concentration n:

$$n = -\frac{B_z J_x}{E_y e} = -\frac{w B_z J_x}{V_H e}$$

or

$$n = -\frac{w B_z I_x}{A V_H e}$$

where A is the cross-sectional area of the specimen.

Since A=w x d (cross-sectional area A = width w times thickness d) we can also write,

$$n = -\frac{B_z I_x}{d V_H e}$$

(i) Magnitude of the Hall voltage. From above,

$$E_y = E_H = -\frac{B_z J_x}{ne}$$

and the Hall voltage is

$$V_y = V_H = -\frac{B_z I_x}{dne}$$

where d is the thickness of the strip, $d = 0.01 \text{ mm} = 1.0\text{x}10^{-5} \text{ m}$

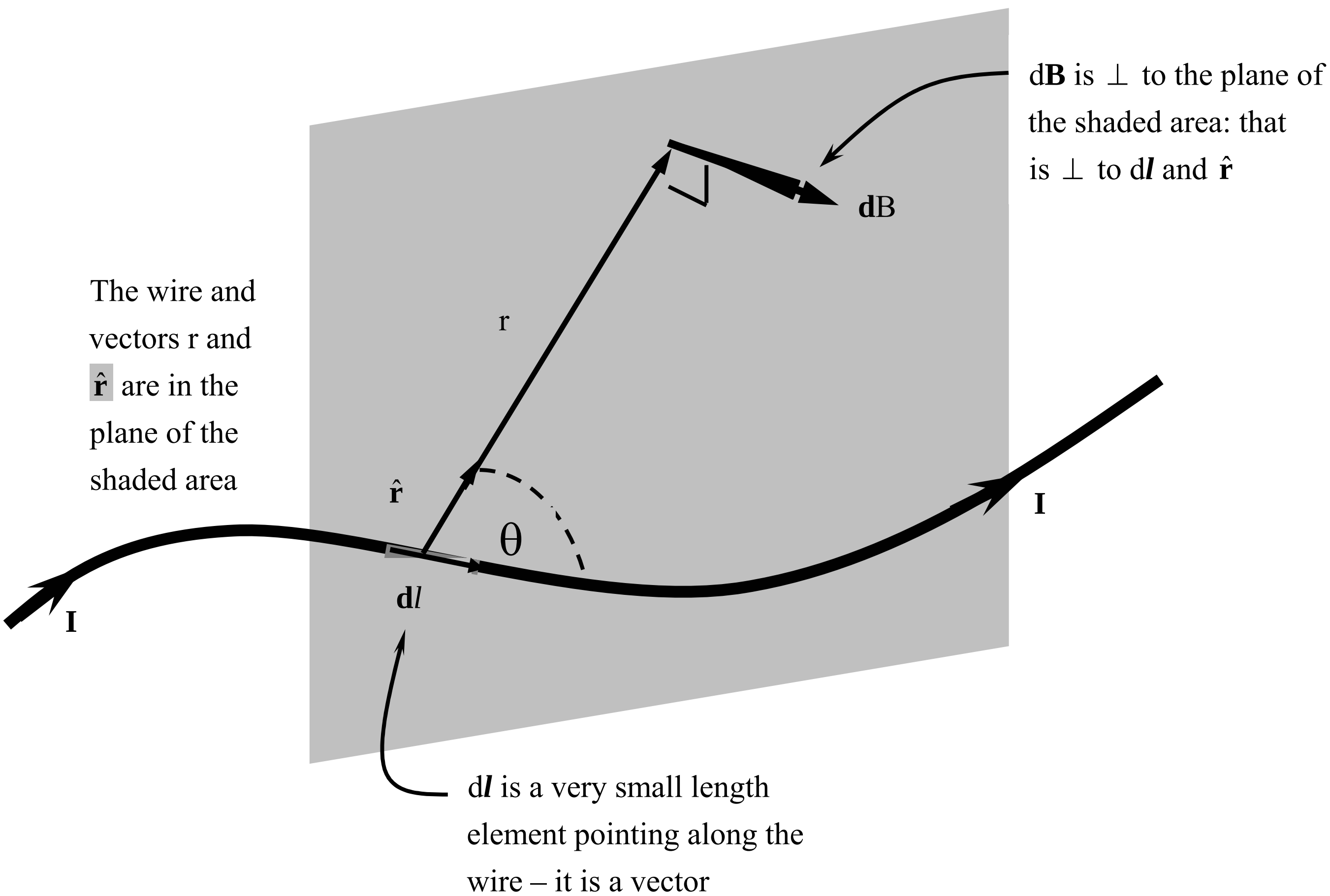
$$V_H = -\frac{B_z I_x}{dne} = \frac{(0.1\text{T})(1.5\text{A})}{(1\text{x}10^{-5}\text{m})(8.5\text{x}10^{28})(1.6\text{x}10^{-19})} = 1.10\text{x}10^{-6}\text{V} = 1.1\,\mu\text{V}$$

Question 5 (Marks)

(a) Biot and Savart were able to write down a quantitative law to calculate B in more general cases where the wire is curved, circular, coiled etc. In differential form the law is

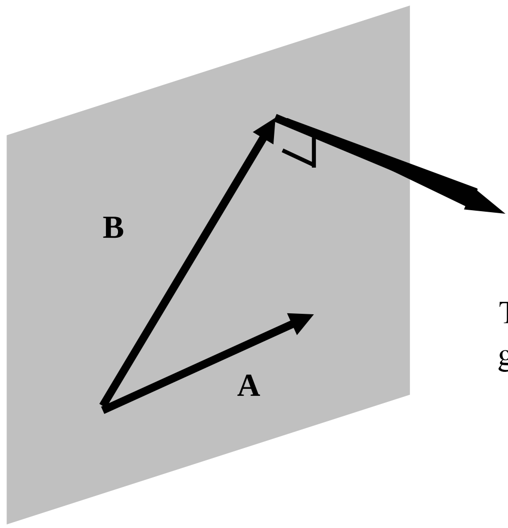
$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}\times\hat{\mathbf{r}}}{r^2}$$

The geometry relevant to this equation is given in the following skectch:



Each length element dl corresponds to a current element $Id\boldsymbol{l}$ which gives a contribution dB to the total magnetic field at a point.

The direction of d**B** is perpendicular to both d***l*** and $\hat{\mathbf{r}}$. It is very handy that the vector cross product gives us precisely what is needed to describe this:*Any* two vectors, say **A** and **B** generate a third vector **C** pointing perpendicular to the plane containing both A and B :



The length of **C** is given by $|C| = |A||B|\sin\theta$

Therefore $d\boldsymbol{l}\mathrm{x}\hat{\mathbf{r}}$ gives the direction of each contribution d**B** from respective current elements Id***l***

The dependence of **B** on distance r from the wire was deduced from experiment (by Pierre Simon Laplace 1749-1827) to be

$$dB \alpha \frac{1}{r^2}$$

The full result for d**B,** including constant (to balance units and provide consistency in the electromagnetic system†) is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}\mathrm{x}\hat{\mathbf{r}}}{r^2}$$

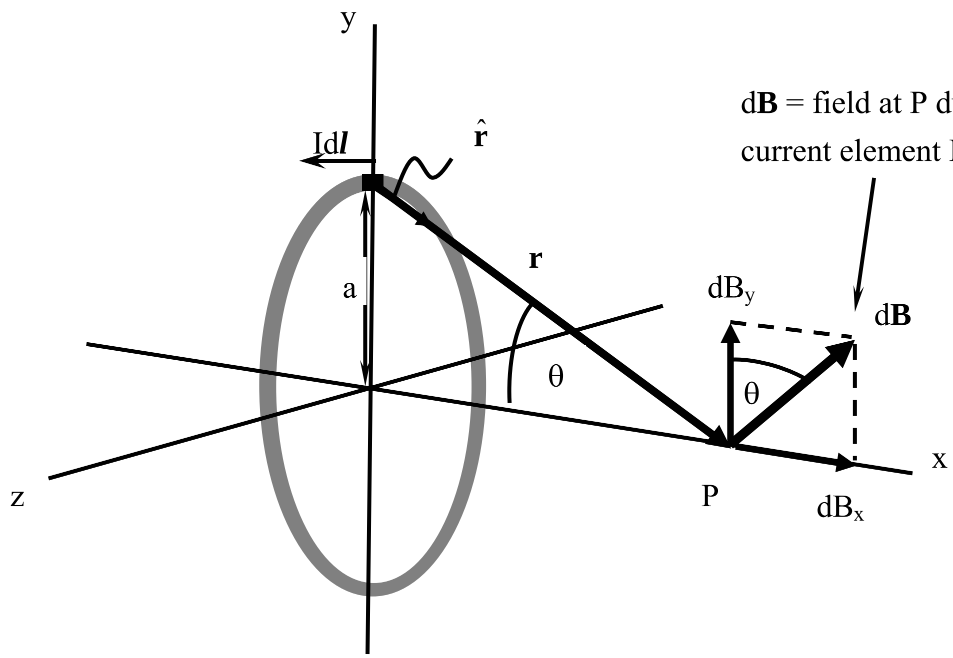
If we ignore the direction (vector) information in the above expression we have

$$dB = \frac{\mu_0}{4\pi}\frac{Id l \sin\theta}{r^2}$$

To get the total B we sum up all the dBs by integrating over the spatial variables (we take the current outside the integral as it is assumed constant):

$$B = \frac{\mu_0 I}{4\pi}\int\frac{dl\sin\theta}{r^2}$$

(b) B on axis for a current-carrying ring or 'current loop'



We want to find the sum of all contributions dB_x at P due to the current elements Id***l*** at all positions around the ring. This is given by the integral around the entire loop. Each contribution to the total field due to element of current Id***l*** is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}\mathrm{x}\hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector (length one unit) pointing from each Id***l*** to point P and $|\mathbf{r}|$ is the distance between each current element and the point P.

Notice that $\hat{\mathbf{r}}$, and therefore **r**, are always perpendicular to Id***l***. This means that

$$Id\boldsymbol{l}\mathrm{x}\hat{r} = Id l \sin\theta = Id l \quad (\sin\theta = \sin 90^\circ = 1)$$

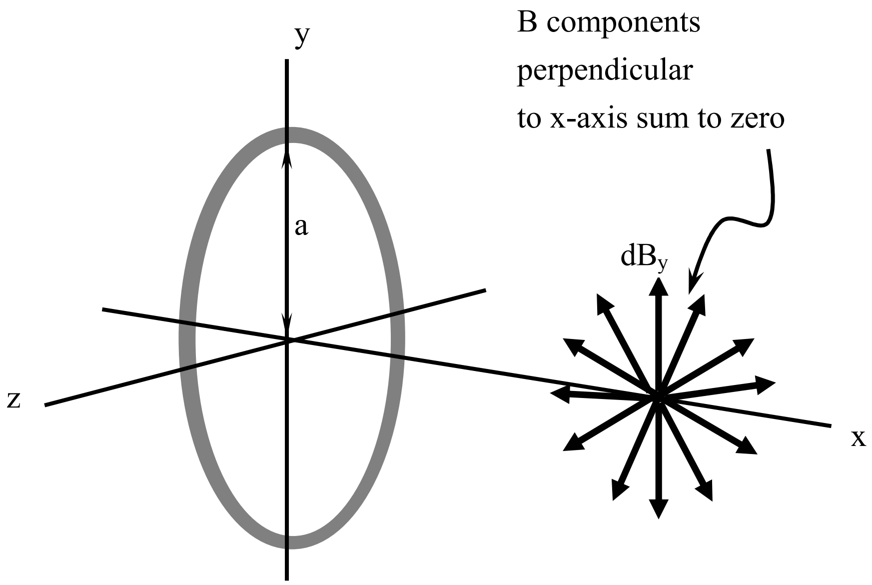
We also see from the geometry that distance r^2 is given by

$$r^2 = a^2 + x^2 \quad \Rightarrow \quad r = \sqrt{a^2 + x^2} \qquad \text{(Pythagoras)}$$

Then,

$$|d\mathbf{B}| = \frac{\mu_0}{4\pi}\frac{I|dl \mathrm{x} \hat{r}|}{r^2} = \frac{\mu_0}{4\pi}\frac{Idl}{[a^2+x^2]}$$

We note that the components of dB along the y-direction, dB_y will sum to zero:



$$dB_x = dB\sin\theta = dB\left(\frac{R}{\sqrt{a^2+x^2}}\right)$$

$$= \frac{\mu_0}{4\pi}\underbrace{\left(\frac{Idl}{[a^2+x^2]}\right)}_{\boxed{r^2 = a^2+x^2}}\underbrace{\left(\frac{R}{\sqrt{a^2+x^2}}\right)}_{\boxed{=\sin\theta}}$$

and the total field in the x-direction (along axis) is

$$B_x = \oint dB_x = \oint \frac{\mu_0}{4\pi}\left(\frac{Idl}{[a^2+x^2]}\right)\left(\frac{a}{\sqrt{a^2+x^2}}\right)$$

$$= \oint \frac{\mu_0}{4\pi}\frac{a}{[a^2+x^2]^{3/2}} Idl$$

Position x, ring radius a and current I are constants in the problem (along with μ_0 and 4π) so that,

$$B_x = \frac{\mu_0 I a}{4\pi\left(a^2 + x^2\right)^{3/2}} \oint dl$$

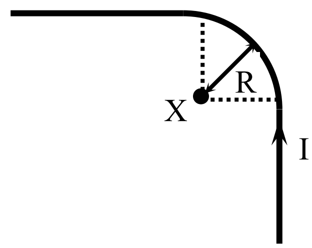
and

$$\oint dl = 2\pi a \qquad \text{(the circumference of the loop)}$$

so that
$$B_x = \frac{\mu_0}{4\pi} \frac{Ia(2\pi a)}{\left(a^2 + x^2\right)^{3/2}}$$

$$= \frac{\mu_0}{2} \frac{Ia^2}{\left(a^2 + x^2\right)^{3/2}}$$

(c) Find the magnetic induction **B** due to the whole wire at the point X located at the centre of the arc, as shown.



The total B-field at X can be found by adding the contribution from each of the two straight sections and the curved section.

Each straight section $\Rightarrow$ equivalent to half an infinitely long straight wire: $B = \frac{1}{2}\left(\frac{\mu_0 I}{2\pi R}\right)$

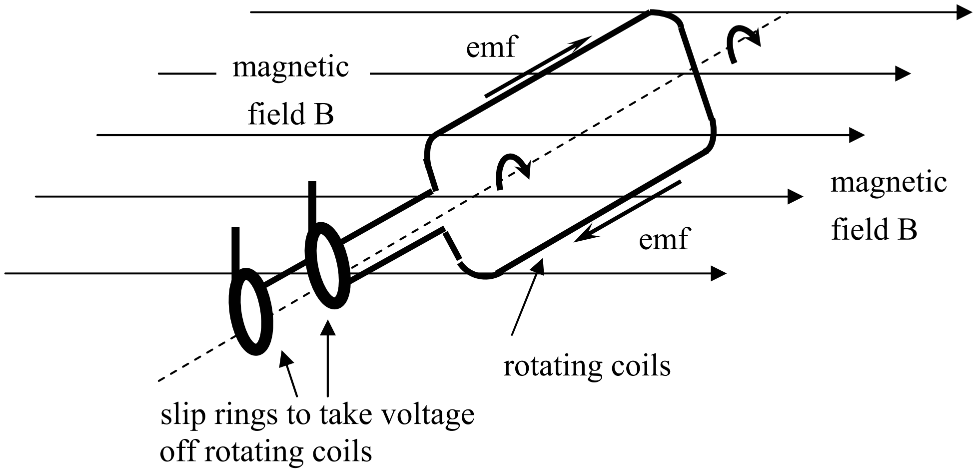
Curved section $\Rightarrow$ equivalent to one quarter of a circular loop: $B = \frac{1}{4}\left(\frac{\mu_0 I}{2R}\right)$

$$\therefore \qquad B_{total} = 2.\frac{1}{2}\left(\frac{\mu_0 I}{2\pi R}\right) + \frac{1}{4}\left(\frac{\mu_0 I}{2R}\right)$$

$$= \left(\frac{1}{2\pi} + \frac{1}{8}\right)\frac{\mu_0 I}{R} = 0.284\left(\frac{\mu_0 I}{R}\right)$$

Question 6 (Marks)

(c) AC generator:



Uniform external magnetic field B tesla applied perpendicular to the long axis of the coils (shown as an open rectangle above). The coils consist of N (many) turns of copper wire.

Conduction electrons in the wire experience a force in the magnetic field B given by

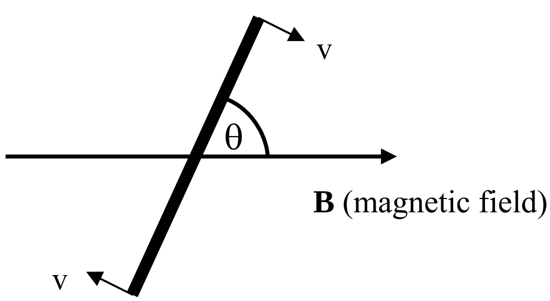
$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

This force on the conduction electrons generates an emf in the coils in the direction indicated in the figure. The emf generated in one of the long sides of the coil has magnitude

$$\text{emf} = \varepsilon = BLv_{\perp}$$

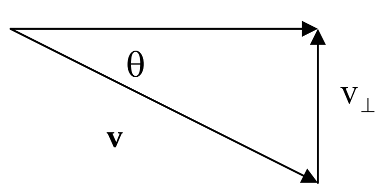
where L is the length of one of the long sides of the coil and $v_{\perp}$ is the perpendicular component of the velocity in the magnetic field.

The coils are rotating in the magnetic field such that end-on they look like:



So that

$$v_{\perp} = v\sin\theta$$



and,

$$\varepsilon = BLv_{\perp}$$
$$= 2BLv\sin\theta \quad \text{(for a single-turn coil).}$$

If the generating coil has N turns we will have

$$\varepsilon = 2NBLv\sin\theta$$

Therefore, as the coils rotate we generate a sinusoidally varying emf.
The tangential speed of the coil v is related to the angular speed ω by

$$v = r\omega$$

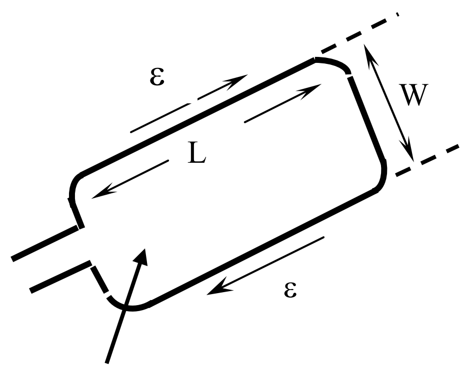
Here we have

$$r = W/2$$

and

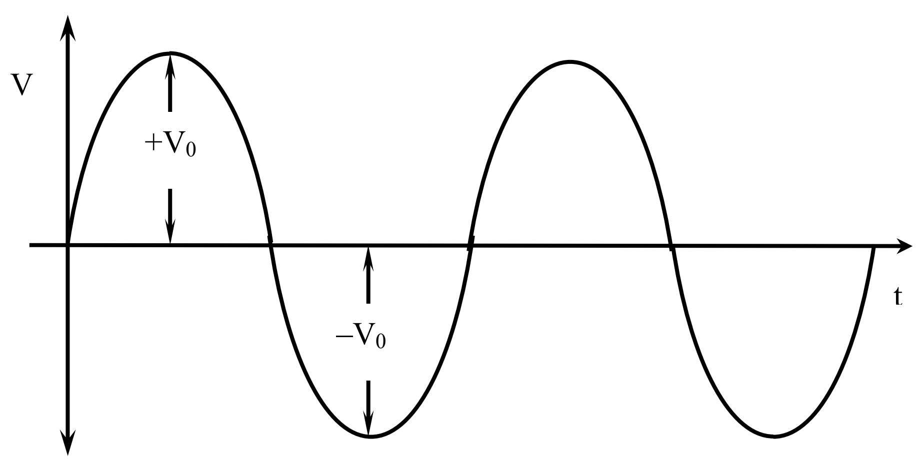
$$\theta = \omega t$$

so that, $\varepsilon = NAB\omega\sin\omega t$



area A = LxW

The voltage output, V(t) of this generator across a load will look like:



(b) $\varepsilon = NAB\omega\sin\omega t$

$N = 400$, $A = (6\text{x}10^{-2})(4\text{x}10^{-2}) = 2.4\text{x}10^{-3}\ \text{m}^2$

$B = \mu H = \mu_0 H = (4\pi\text{x}10^{-7})(100) = 400\pi\text{x}10^{-7}\ \text{T T}$

$\omega = 2\pi f = 2\pi\dfrac{600}{60} = 20\pi\ \text{rads}^{-1}$

$$\varepsilon = (400)(2.4\text{x}10^{-3})(400\pi\text{x}10^{-7})(20\pi)\sin\omega t$$
$$\varepsilon = (7.68\text{x}10^{-4})\pi^{2}\sin\omega t = (7.58\text{x}10^{-3})\sin\omega t$$

and the r.m.s. value is

$$\varepsilon_{rms} = \frac{\varepsilon_{peak}}{\sqrt{2}} = \frac{7.58\text{x}10^{-3}}{\sqrt{2}} = 5.36\text{x}10^{-3}\,\text{V} = 5.36\text{ mV}$$

(c) Faraday's law is

$$\varepsilon = -\frac{d\Phi}{dt} \text{ (for a single turn coil) where } \Phi = \mathbf{B}.\mathbf{A}$$

$$A = (6\text{x}10^{-2})(4\text{x}10^{-2}) = 2.4\text{x}10^{-3}\ \text{m}^{2}$$

The magnetic flux is

$$\Phi = \mathbf{B}.\mathbf{A} = \mu_0\mathbf{H}.\mathbf{A} = (4\pi\text{x}10^{-7})(100)(2.4\text{x}10^{-3}).\cos\theta = (9.6\text{x}10^{-8}\pi)\cos\theta$$

where θ is angle between **H** and **A**, and

$$\theta = \omega t = 2\pi f t$$

and

$$\varepsilon = -\frac{d\Phi}{dt} = (400)(9.6\text{x}10^{-8}\pi)(20\pi)\sin(2\pi f t)$$

$$= (7.58\text{x}10^{-3})\sin 2\pi f t$$

$$|\varepsilon_{rms}| = \frac{7.58\text{x}10^{-3}}{\sqrt{2}}\ \text{V} = 5.36\text{x}10^{-3}\ \text{V} = 5.36\text{ mV}$$