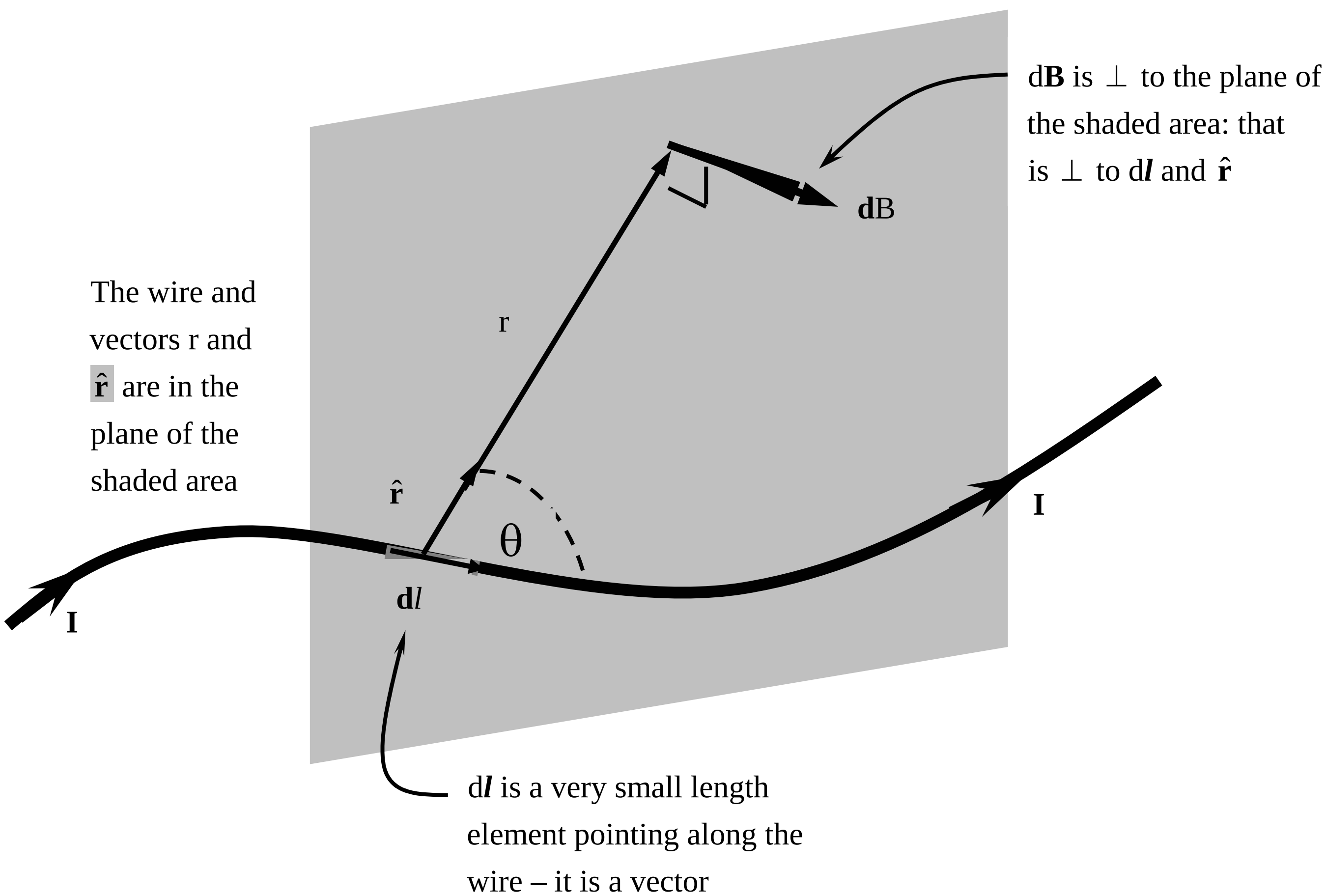
Question 5 (Marks)

(a) Biot and Savart were able to write down a quantitative law to calculate B in more general cases where the wire is curved, circular, coiled etc. In differential form the law is

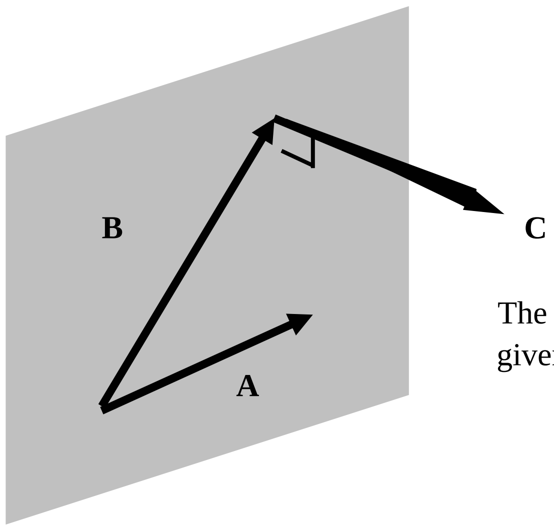
$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}x\hat{\mathbf{r}}}{r^2}$$

The geometry relevant to this equation is given in the following skectch:



Each length element dl corresponds to a current element $Id\boldsymbol{l}$ which gives a contribution dB to the total magnetic field at a point.

The direction of d**B** is perpendicular to both d***l*** and $\hat{\mathbf{r}}$. It is very handy that the vector cross product gives us precisely what is needed to describe this:*Any* two vectors, say **A** and **B** generate a third vector **C** pointing perpendicular to the plane containing both A and B :



The length of **C** is given by $|C| = |A||B|\sin\theta$

Therefore $d\boldsymbol{l} \times \hat{\mathbf{r}}$ gives the direction of each contribution d**B** from respective current elements Id***l***

The dependence of **B** on distance r from the wire was deduced from experiment (by Pierre Simon Laplace 1749-1827) to be

$$dB \alpha \frac{1}{r^2}$$

The full result for d**B,** including constant (to balance units and provide consistency in the electromagnetic system†) is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{Id\boldsymbol{l}\times\hat{\mathbf{r}}}{r^2}$$

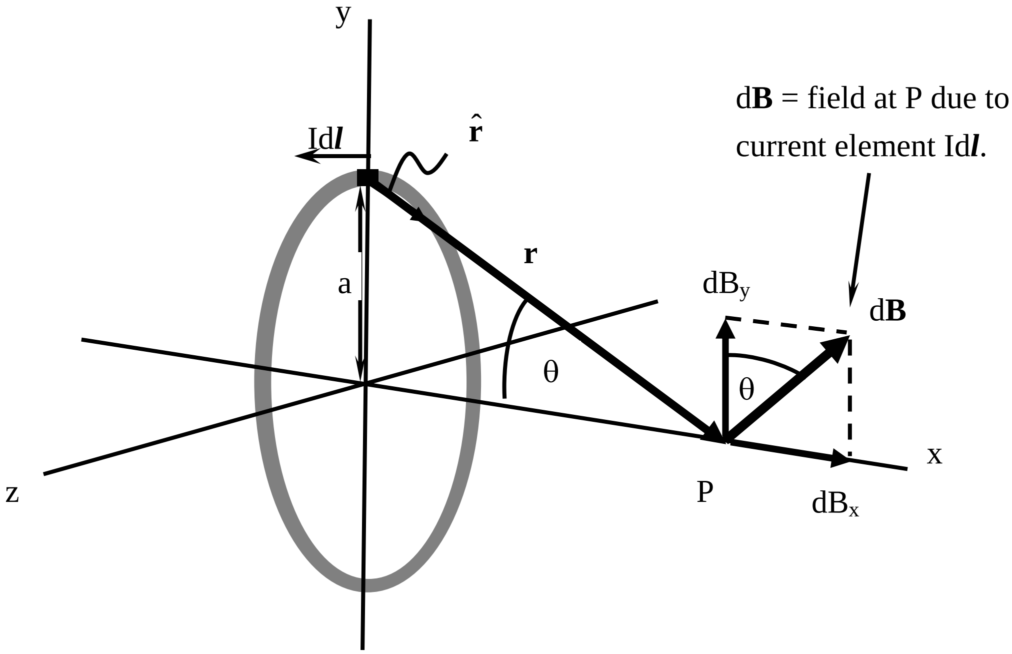
If we ignore the direction (vector) information in the above expression we have

$$dB = \frac{\mu_0}{4\pi}\frac{Id l \sin\theta}{r^2}$$

To get the total B we sum up all the dBs by integrating over the spatial variables (we take the current outside the integral as it is assumed constant):

$$B = \frac{\mu_0 I}{4\pi}\int\frac{dl\sin\theta}{r^2}$$

(b) B on axis for a current-carrying ring or ‘current loop’



We want to find the sum of all contributions dB_x at P due to the current elements Id***l*** at all positions around the ring. This is given by the integral around the entire loop. Each contribution to the total field due to element of current Id***l*** is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I d\boldsymbol{l} x \hat{\mathbf{r}}}{r^2}$$

where $\hat{\mathbf{r}}$ is the unit vector (length one unit) pointing from each Id***l*** to point P and $|\mathbf{r}|$ is the distance between each current element and the point P.

Notice that $\hat{\mathbf{r}}$, and therefore **r**, are always perpendicular to Id***l***. This means that

$$I d\boldsymbol{l} x \hat{r} = I dl \sin\theta = I dl \quad (\sin\theta = \sin 90^\circ = 1)$$

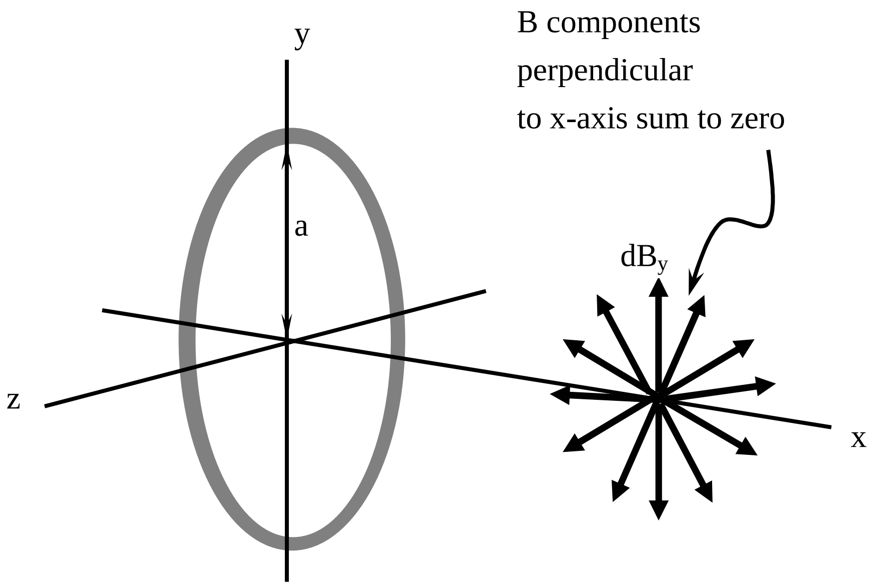
We also see from the geometry that distance r^2 is given by

$$r^2 = a^2 + x^2 \quad \Rightarrow \quad r = \sqrt{a^2 + x^2} \qquad \text{(Pythagoras)}$$

Then,

$$|d\mathbf{B}| = \frac{\mu_0}{4\pi}\frac{I|dl\mathrm{x}\hat{r}|}{r^2} = \frac{\mu_0}{4\pi}\frac{Idl}{[a^2+x^2]}$$

We note that the components of dB along the y-direction, dB_y will sum to zero:



$$dB_x = dB\sin\theta = dB\left(\frac{R}{\sqrt{a^2+x^2}}\right)$$

$$= \frac{\mu_0}{4\pi}\underbrace{\left(\frac{Idl}{[a^2+x^2]}\right)}_{\boxed{r^2 = a^2+x^2}}\underbrace{\left(\frac{R}{\sqrt{a^2+x^2}}\right)}_{\boxed{=\sin\theta}}$$

and the total field in the x-direction (along axis) is

$$B_x = \oint dB_x = \oint \frac{\mu_0}{4\pi}\left(\frac{Idl}{[a^2+x^2]}\right)\left(\frac{a}{\sqrt{a^2+x^2}}\right)$$

$$= \oint \frac{\mu_0}{4\pi}\frac{a}{[a^2+x^2]^{3/2}}Idl$$

Position x, ring radius a and current I are constants in the problem (along with μ_0 and 4π) so that,

$$B_x = \frac{\mu_0 Ia}{4\pi\left(a^2 + x^2\right)^{3/2}} \oint dl$$

and

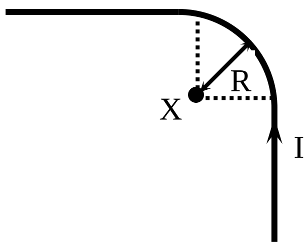
$$\oint dl = 2\pi a \qquad \text{(the circumference of the loop)}$$

so that

$$B_x = \frac{\mu_0}{4\pi} \frac{Ia(2\pi a)}{\left(a^2 + x^2\right)^{3/2}}$$

$$= \frac{\mu_0}{2} \frac{Ia^2}{\left(a^2 + x^2\right)^{3/2}}$$

(c) Find the magnetic induction **B** due to the whole wire at the point X located at the centre of the arc, as shown.



The total B-field at X can be found by adding the contribution from each of the two straight sections and the curved section.

Each straight section $\Rightarrow$ equivalent to half an infinitely long straight wire: $B = \frac{1}{2}\left(\frac{\mu_0 I}{2\pi R}\right)$

Curved section $\Rightarrow$ equivalent to one quarter of a circular loop: $B = \frac{1}{4}\left(\frac{\mu_0 I}{2R}\right)$

$$\therefore \qquad B_{total} = 2.\frac{1}{2}\left(\frac{\mu_0 I}{2\pi R}\right) + \frac{1}{4}\left(\frac{\mu_0 I}{2R}\right)$$

$$= \left(\frac{1}{2\pi} + \frac{1}{8}\right)\frac{\mu_0 I}{R} = 0.284\left(\frac{\mu_0 I}{R}\right)$$