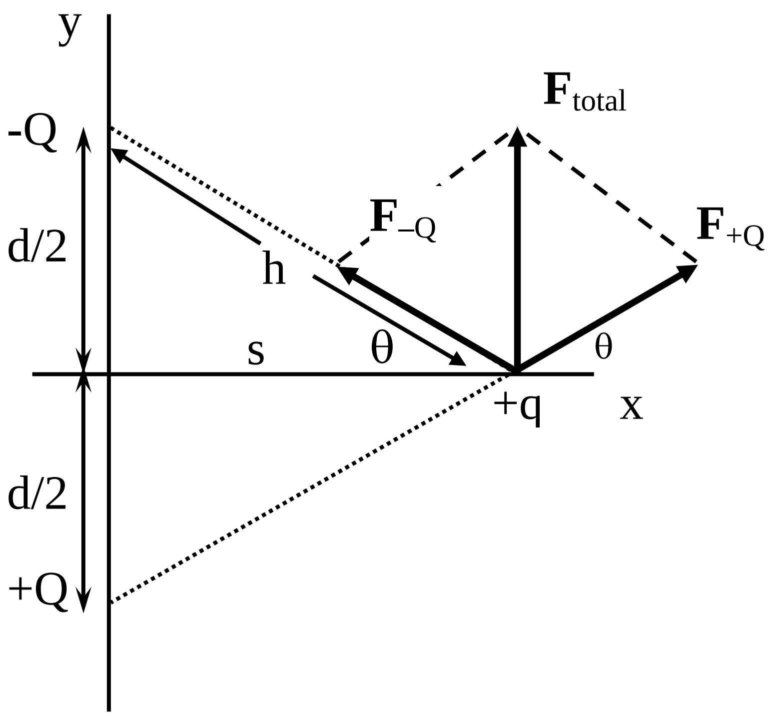
Question 1 (Marks 15)

(a)



The distance s is given by

$$s = \sqrt{h^2 + (a/2)^2}$$

Coulomb's law for charges q_1 and q_2 is

$$F = k_0 \frac{q_1 q_2}{r^2}$$

which gives the magnitude of the force on +q due to +Q and –Q

$$\left|\mathbf{F}_{+Q}\right| = \left|\mathbf{F}_{-Q}\right| = k_0 \frac{qQ}{h^2} = k_0 \frac{qQ}{\left[s^2 + (a/2)^2\right]}$$

and the magnitude of the total force is

$$\left|\mathbf{F}_{\text{total}}\right| = 2k_0 \frac{qQ}{h^2}\sin\theta = 2k_0 \frac{qQ}{\left[s^2 + (a/2)^2\right]}\sin\theta$$

substituting $\sin\theta = \dfrac{a/2}{h} = \dfrac{a/2}{\sqrt{s^2 + (a/2)^2}}$

$$\left|\mathbf{F}_{\text{total}}\right| = 2k_0 \frac{qQ}{\left[s^2 + (a/2)^2\right]} \frac{a/2}{\sqrt{s^2 + (a/2)^2}}$$

$$\left|\mathbf{F}_{\text{total}}\right| = k_0 \frac{aqQ}{\left[s^2 + (a/2)^2\right]^{3/2}}$$

By symmetry see that $\mathbf{F}_{\text{total}}$ is parallel to the y-axis, in $\hat{\mathbf{j}}$ direction so that

$$\left|\mathbf{F}_{\text{total}}\right| = k_0 \frac{aqQ}{\left[s^2 + (a/2)^2\right]^{3/2}}\hat{\mathbf{j}}$$

(b) The magnitude of the force between two charges q_1, q_2 is given by Coulomb's law:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \qquad (1)$$

where ε_0 is the permittivity of free space. The force can be seen to arise from the electric field around charge q_i

$$E = \frac{F}{q} = \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r^2} \qquad (2)$$

If the charges are immersed in a medium other than free space, ε_0 is replaced in expressions (1) and (2) by the permittivity value ε of that medium. Thus, permittivity is a scaling factor for electric field strength in a medium and quantifies the electrostatic behaviour of the medium.

(c) Dielectric materials with high permittivity values are used in the production of capacitors. For example, in a parallel plate capacitor, the electric field between the plates carrying surface charge density σ is

$$E = \frac{\sigma}{\varepsilon}$$

a dielectric material with permittivity $\varepsilon >> 1$ inserted between the plates lowers the electric field between the plates, thus allowing greater charge to be stored at a given potential difference, i.e. a greater value of C (farads) is obtained for a given geometry.

Other possible engineering examples: dielectric in a coaxial cable; the SiO_2 layer in CMOS microelectronics etc.

QUESTION 2 (Marks 16)

(a) The charge density varies with radius given by

$$\rho(r) = -\frac{5\rho_0}{2}\left[1 - \frac{r^2}{R^2}\right]$$

where $\rho_0 = \dfrac{Q}{(4/3\pi R^3)}$.

(i) The volume of a spherical shell of charge of thickness dr' at radius r' away from the nucleus the is

$$dV = 4\pi r'^2 dr'$$

and the charge in this shell is

$$dq = \rho(r')dV = 4\pi r'^2 \rho(r')dr'$$

Integrating from $r' = 0 \text{to} r' = r$,

$$q(r) = \int dq = \int_0^r 4\pi\rho(r')r'^2 dr' = 4\pi\int_0^r -\frac{5\rho_0}{2}\left(1 - \frac{r'^2}{R^2}\right)r'^2 dr'$$

$$= -10\pi\rho_0\left[\frac{1}{3}r'^3 - \frac{r'^5}{5R^2}\right]_0^r$$

$$= -10\pi\left(\frac{Q}{4\pi R^3/3}\right)\left(\frac{1}{3}r^3 - \frac{r^5}{5R^2}\right)$$

$$= -\frac{5r^3Q}{2R^2} + \frac{3r^5}{2R^5}$$

The total charge as a fn. of position is charge on nucleus Q plus the electronic contribution:

$$q(r)_{total} = Q_{nuc} + q(r) = Q + \left(-\frac{5r^3 Q}{2R^2} + \frac{3r^5}{2R^5} \right)$$

$$q(r)_{total} = Q_{nuc} + q(r) = Q\left(1 - \frac{5r^3}{2R^2} + \frac{3r^5}{2R^5} \right)$$

The E-field around the nucleus is symmetric: $\mathbf{E} = E(r)\hat{\mathbf{r}}$. Gauss' law gives

$$\int_{surface} \mathbf{E}.d\mathbf{A} = \int_{surface} E(r)\hat{\mathbf{r}}.dA\hat{\mathbf{r}} = \int_{surface} E(r)dA = E(r) \int_{surface} dA = E(r)(4\pi r^2) = \frac{\Sigma q}{\varepsilon_0}$$

(ii) In the region $0 < r < R$,

$$E(r) = k_0 \frac{q(r)}{r^2} = k_0 \frac{q}{r^2}\left(1 - \frac{5r^3}{2R^2} + \frac{3r^5}{2R^5} \right)$$

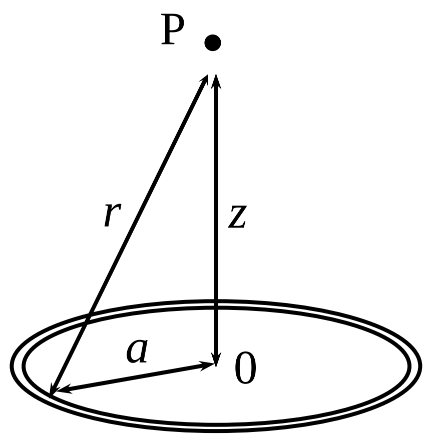
At radius $r = R$ the charge is

$$q_R = Q\left(1 - \frac{5r^3}{2R^2} + \frac{3r^5}{2R^5} \right) = 0$$

as it must be for a neutral atom. By Gauss' law therefore, we must have $E(r) = 0$ for $r \geq R$.

Question 3 (Marks 10)

The diagram shows a circular ring of uniform electric charge of radius *a*. The total charge on the ring is Q coulombs. Derive an expression in terms of *z* and *a* for the electric potential at a point P vertically above the centre of the ring, 0, as shown.



The potential at point P with position vector **r** is

$$V = k_0 \int \frac{dq}{r}$$

where $k_0 = \frac{1}{4\pi\varepsilon_0}$. We note from the geometry that

$$r = \sqrt{a^2 + z^2}$$

and then

$$V = k_0 \int \frac{dq}{r} = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{\sqrt{a^2 + z^2}}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{a^2 + x^2}}$$

$$|\varepsilon_{rms}| = \frac{7.58x10^{-3}}{\sqrt{2}} V = 5.36x10^{-3} V = 5.36 mV$$