Qu_1.

a) $Q = p + iaq$ and $P = (p - iaq)/2ia$ so

$$[Q,P] = ia \times \frac{1}{2ia} - 1\left(-\frac{1}{2}\right) = 1$$

b) Find $F_2(q,P)$. Need $p = p(q,P)$ and $Q = Q(q,P)$.

$$Q = p + iaq \qquad (1)$$
$$2iaP = p - iaq \qquad (2)$$

(1)-(2) gives

$$Q - 2iaP = 2iaq \quad \Rightarrow \quad Q = 2ia(P + q) \qquad (3)$$

then

$$p = Q - iaq = 2ia(P + q) - iaq = ia(2P + q) \qquad (4)$$

$$\begin{aligned} F_2 &= \int p(q,P)dq + f(P) \\ &= ia\int (2P + q)dq + f(P) \\ &= ia\left(2Pq + \tfrac{1}{2}q^2\right) + f(P) \end{aligned} \qquad \begin{aligned} F_2 &= \int Q(q,P)dP + g(q) \\ &= 2ia\int (P + q)dq + g(q) \\ &= ia\left(P^2 + 2Pq\right) + g(q) \end{aligned}$$

comparing

$$F_2(q,P) = ia\left(\tfrac{1}{2}q^2 + 2qP + P^2\right)$$

c) Note that $2iaPQ = (p - iaq)(p + iaq) = p^2 + a^2q^2$, so

$$H = \tfrac{1}{2}\left(p^2 + a^2q^2\right) = K(Q,P) = iaPQ$$

d) $\dot{Q} = \partial K/\partial P = iaQ$ and $\dot{P} = -\partial K/\partial Q = -iaP$

e) Solutions are $Q(t) = Q_0e^{iat}$ and $P(t) = P_0e^{-iat}$

f) Chain rule $du = \sum\left(\frac{\partial u}{\partial q}dq + \frac{\partial u}{\partial p}dp\right) + \frac{\partial u}{\partial t}dt$

$$\frac{du}{dt} = \sum\left(\frac{\partial u}{\partial q}\dot{q} + \frac{\partial u}{\partial p}\dot{p}\right) + \frac{\partial u}{\partial t} = \sum\left(\frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial u}{\partial p}\frac{\partial H}{\partial q}\right) + \frac{\partial u}{\partial t} = [u,H] + \frac{\partial u}{\partial t}$$

g) Then $\dot{q} = [q,H]$ and $\dot{p} = [p,H]$

Qu_2. Let $x = a + b$ then the translational kinetic energy is $T_{CM} = \frac{m}{2}(\dot{x}^2 + x^2\dot{\phi}^2)$. When ϕ goes from zero to some non-zero value, the small sphere rotates thru an angle of $\phi + \theta$ with respect to the vertical line, hence is rotation rate $\omega = \dot{\theta} + \dot{\phi}$. The rotational kinetic energy is

$$T_{rot} = \tfrac{1}{2}I\omega^2 = \tfrac{1}{2}\tfrac{2}{5}ma^2(\dot{\theta} + \dot{\phi})^2 \quad \text{and} \quad V = mgx\cos\phi$$

Lagrange's equations $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_l a_{il}\lambda_l$$

Constraints: $dx = 0$ small sphere remains attached

$bd\phi - ad\theta = 0$ no slipping

$$L = \tfrac{1}{2}m(\dot{x}^2 + x^2\dot{\phi}^2) + \tfrac{1}{5}ma^2(\dot{\theta} + \dot{\phi})^2 - mgx\cos\phi$$

x: $$\frac{d}{dt}\left(m\dot{x}\right) - \left(mx\dot{\phi}^2 - mg\cos\phi\right) = \lambda_1 \quad (1)$$

ϕ: $$\frac{d}{dt}\left(mx^2\dot{\phi} + \tfrac{2}{5}ma^2(\dot{\theta} + \dot{\phi})\right) - \left(mgx\sin\phi\right) = b\lambda_2 \quad (2)$$

θ: $$\frac{d}{dt}\left(\tfrac{2}{5}ma^2(\dot{\theta} + \dot{\phi})\right) = -a\lambda_2 \quad (3)$$

from (2) and (3)

$$mx^2\ddot{\phi} + \tfrac{2}{5}ma^2(\ddot{\theta} + \ddot{\phi}) - mgx\sin\phi = -\tfrac{b}{a}\tfrac{2}{5}ma^2(\ddot{\theta} + \ddot{\phi})$$

but $bd\phi - ad\theta = 0$ implies $\ddot{\theta} = \frac{b}{a}\ddot{\phi}$ so that $a^2(\ddot{\theta} + \ddot{\phi}) = a(a+b)\ddot{\phi}$

$$mx^2\ddot{\phi} + \tfrac{2}{5}ma(a+b)^2\ddot{\phi} - mgx\sin\phi = 0$$

but $x = a + b$, so $mx\ddot{\phi} + \tfrac{2}{5}mx\ddot{\phi} - mg\sin\phi = 0 \quad \Rightarrow \quad \tfrac{7}{5}mx\ddot{\phi} - mg\sin\phi = 0$

multiply thru by $\dot{\phi}$ and integrate with respect to time

$$\frac{d}{dt}\left\{\tfrac{7}{5}mx\tfrac{1}{2}\dot{\phi}^2 + mg\cos\phi\right\} = 0 \quad \text{so} \quad \tfrac{7}{10}x\dot{\phi}^2 + g\cos\phi = C$$

Initial conditions are $\phi = 0$, $\dot{\phi} = 0$ so $$\dot{\phi}^2 = \frac{10g}{7x}\left(1 - \cos\phi\right)$$

From (1) $\lambda_1 = mg\cos\phi - mx\dot{\phi}^2 = \frac{mg}{7}\left(17\cos\phi - 10\right)$. So $\lambda_1 = 0$, when $\cos\phi = \frac{10}{17}$.