THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

NOVEMBER 2010

PHYS3510

Advanced Mechanics, Fields and Chaos

Time Allowed – 2 hours

Total number of questions - 4

Answer ALL questions

All questions ARE of equal value

Candidates may bring their own approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

FORMULA SHEET

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk} = Q_k$$

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad T = \tfrac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Spherical polar coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \qquad T = \tfrac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right) \\ z &= r\cos\theta \end{aligned}$$

Canonical Transformations

1) $F_1(q,Q,t)$ $\quad p_i = \dfrac{\partial F_1}{\partial q_i}$ $\quad P_i = -\dfrac{\partial F_1}{\partial Q_i}$ $\quad K = H + \dfrac{\partial F_1}{\partial t}$

2) $F_2(q,P,t)$ $\quad p_i = \dfrac{\partial F_2}{\partial q_i}$ $\quad Q_i = \dfrac{\partial F_2}{\partial P_i}$ $\quad K = H + \dfrac{\partial F_2}{\partial t}$

3) $F_3(p,Q,t)$ $\quad q_i = -\dfrac{\partial F_3}{\partial p_i}$ $\quad P_i = -\dfrac{\partial F_3}{\partial Q_i}$ $\quad K = H + \dfrac{\partial F_3}{\partial t}$

4) $F_4(p,P,t)$ $\quad q_i = -\dfrac{\partial F_4}{\partial p_i}$ $\quad Q_i = \dfrac{\partial F_4}{\partial P_i}$ $\quad K = H + \dfrac{\partial F_4}{\partial t}$

Poisson Bracket $\quad [u,v]_{q,p} = \sum_k \left(\dfrac{\partial u}{\partial q_k}\dfrac{\partial v}{\partial p_k} - \dfrac{\partial u}{\partial p_k}\dfrac{\partial v}{\partial q_k}\right)$

Hamilton-Jacobi Theory	$H(q,p,t)$	$H(q,p) = \text{constant}$
Canonical transformation to	$Q_i,\ P_i$	P_i (constant of motion)
New Hamiltonian	$K = 0$	$K = H(P_i) = \alpha_1$
New equations of motion	$\dot{Q}_i = \dfrac{\partial K}{\partial P_i} = 0$ $\dot{P}_i = -\dfrac{\partial K}{\partial Q_i} = 0$	$\dot{Q}_i = \dfrac{\partial K}{\partial P_i} = \nu_i$ $\dot{P}_i = -\dfrac{\partial K}{\partial Q_i} = 0$

With solutions	$Q_i = \beta_{i,} \quad P_i = \gamma_i$	$Q_i = \nu_i t + \beta_{i,} \quad P_i = \gamma_i$
Generating Function	Hamilton's Principle $S(q,P,t)$	Hamilton's Characteristic $W(q,P)$
Hamilton-Jacobi equation	$H\left(q_i,\dfrac{\partial S}{\partial q_i},t\right)+\dfrac{\partial S}{\partial t}=0$	$H\left(q_i,\dfrac{\partial W}{\partial q_i}\right)-\alpha_1=0$
New constant momenta (one choice)	$P_i = \gamma_i(\alpha_1,\ldots,\alpha_n)$ $\gamma_i = \alpha_i$	$P_i = \gamma_i(\alpha_1,\ldots,\alpha_n)$ $\gamma_i = \alpha_i$
Hamilton-Jacobi solution	$S = S(q_i,\gamma_i,t)$	$W = W(q_i,\gamma_i)$
First half of transformation	$p_i = \dfrac{\partial S}{\partial q_i}$	$p_i = \dfrac{\partial W}{\partial q_i}$
Second half of transformation	$Q_i = \dfrac{\partial S}{\partial \gamma_i} = \beta_i$	$Q_i = \dfrac{\partial W}{\partial \gamma_i} = \nu_i(\gamma_j)t + \beta_i$

Action-angle Variables $\quad J_i = \oint p_i dq_i = \oint \dfrac{\partial W_i(q_i,\alpha_1,\ldots,\alpha_n)}{\partial q_i} dq_i \qquad w_i = \dfrac{\partial W}{\partial J_i}$

Euler-Lagrange equation for fields $\quad \dfrac{\partial \mathfrak{L}}{\partial \eta} - \dfrac{d}{dt}\left(\dfrac{\partial \mathfrak{L}}{\partial \dot{\eta}}\right) - \dfrac{d}{dx}\left(\dfrac{\partial \mathfrak{L}}{\partial(\partial \eta/\partial x)}\right) = 0$

Mathematical identities

$$\sin^2 Q + \cos^2 Q = 1 \qquad \tan^2 Q + 1 = \sec^2 Q$$

$$1 + \cos 2\phi = 2\cos^2 \phi \qquad 1 - \cos 2\phi = 2\sin^2 \phi$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = \operatorname{cosec}^2 x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

QUESTION 1. (20 marks)

A particle of mass m slides without friction on the inside of a hollow spherical shell of radius a under the influence of gravity.

(a) Use the method of Lagrange multipliers to obtain the equations of motion for the system in spherical polar coordinates.

(b) What are the constants of the motion?

(c) If the particle starts at the bottom of the sphere with initial velocity v (at $t = 0$, $\theta = \pi$ and $\dot{\theta} = v/a$), identify three different types of motion of the particle.

(d) If the particle falls off the interior of the shell, find the value of θ at which this happens for a given initial velocity v.

QUESTION 2. (20 marks)

a) For the Hamiltonian

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right),$$

find a canonical transformation that produces the new Hamiltonian $K = \frac{1}{2}\left(Q^2 + P^2\right)$, and prove that it is canonical.

b) Find the $F_2(q,P)$ generating function that generates this canonical transformation.

c) If the generating function $F_2(q,P) = \sum q_i P_i + \varepsilon G(q,P)$ generates an infinitesimal contact transformation, find the change in q and p.

d) If $G \equiv L_z = (r_i \times p_i)_z = \sum_i \left(x_i p_{yi} - y_i p_{xi}\right)$ where L_z is the z-component of angular momentum, determine the physical nature of the infinitesimal contact transformation generated by L_z.

QUESTION 3. (20 marks)

a) If the Hamiltonian for a particle of mass m sliding on a cycloid

$$x = C(\theta - \sin\theta)$$
$$y = C(1 + \cos\theta)$$

is given by

$$H = \frac{p_\theta^2}{8mC^2 \sin^2(\theta/2)} + 2Cmg\cos^2(\theta/2)$$

use action-angle variables to show that the frequency of the motion is independent of the energy.

Note that:
$$1 - \cos\theta = 2\sin^2\tfrac{\theta}{2}$$
$$1 + \cos\theta = 2\cos^2\tfrac{\theta}{2}$$

b) If the Lagrangian density for displacements of an elastic rod is given by

$$L = \tfrac{1}{2}\left(\mu\dot{\eta}^2 - Y\left(\frac{\partial\eta}{\partial x}\right)^2\right)$$

find the Lagrangian equations of motion. What is the physical meaning of this result?

QUESTION 4. (20 marks)

a) Determine the fixed points and calculate their stability properties for the equations

$$\dot{x} = x - xy. \qquad \dot{y} = xy - y$$

b) The stability of an iterative mapping $x_{n+1} = f(x_n)$ can be determined by calculating the Lyapunov exponent defined by

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |f'(x_i)|$$

Find the Lyapunov exponent as a function of μ for the two fixed points of the quadratic map, $x_{n+1} = \mu x_n(1 - x_n)$. Discuss the stability of the fixed points as $\mu \to 2$?

c) If the derivative of two applications of the quadratic map at the 2-cycle is $f_\mu^{2'} = 4 + 2\mu - \mu^2$, explain the behaviour of the Lyapunov exponent as $\mu \to 2$ from above, and also as μ increases from 2.

d) What are tangent bifurcations and pitchfork bifurcations and how do they arise?

e) For the quadratic map we can describe the type of bifurcations that lead to cycles of a particular length. Complete all the missing entries in the table.

cycle length	periodic points	'prime' points	number of n-cycles	created by	created by
n	2^n			tangent	pitchfork
1	2	2	2	1	0
2	4	2	1	0	1
3	8	6	2	1	0
4	16				
5	32				
6	64				
7	128				
8	256				

f) Define the unstable manifold of a fixed point. Prove that unstable manifolds from different fixed points do not intersect.