THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

PHYS 3510 ADVANCED MECHANICS, FIELDS AND CHAOS

MID-SESSION TEST - 11 SEPTEMBER 2008

Do both questions.

Both questions of equal marks.

FORMULA SHEET

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_l^m \lambda_l a_{lk} = Q_k$$

Canonical Transformations

1) $F_1(q,Q,t)$ $\quad p_i = \frac{\partial F_1}{\partial q_i}$ $\quad P_i = -\frac{\partial F_1}{\partial Q_i}$ $\quad K = H + \frac{\partial F_1}{\partial t}$

2) $F_2(q,P,t)$ $\quad p_i = \frac{\partial F_2}{\partial q_i}$ $\quad Q_i = \frac{\partial F_2}{\partial P_i}$ $\quad K = H + \frac{\partial F_2}{\partial t}$

3) $F_3(p,Q,t)$ $\quad q_i = -\frac{\partial F_3}{\partial p_i}$ $\quad P_i = -\frac{\partial F_3}{\partial Q_i}$ $\quad K = H + \frac{\partial F_3}{\partial t}$

4) $F_4(p,P,t)$ $\quad q_i = -\frac{\partial F_4}{\partial p_i}$ $\quad Q_i = \frac{\partial F_4}{\partial P_i}$ $\quad K = H + \frac{\partial F_4}{\partial t}$

Poisson Bracket $\quad [u,v]_{q,p} = \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}\right)$

Mathematical identities

$$\sin^2 Q + \cos^2 Q = 1 \qquad \tan^2 Q + 1 = \sec^2 Q \qquad 1 + \cos 2\phi = 2\cos^2\phi \qquad 1 - \cos 2\phi = 2\sin^2\phi$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = \operatorname{cosec}^2 x \qquad \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

QUESTION 1. (7.5 marks)

a) Show that the transformation

$$Q = \tan^{-1}\left(\frac{\alpha q}{p}\right) \quad \text{and} \quad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right)$$

is canonical. Rearrange the transformation equations to obtain

$$q = \frac{p}{\alpha}\tan Q \quad \text{and} \quad P = \frac{p^2}{2\alpha}\sec^2 Q.$$

b) Find the generating function for this transformation of type $F_3(p,Q)$ using both of the relations

$$q = -\frac{\partial F_3(p,Q)}{\partial p} \quad \text{and} \quad P = -\frac{\partial F_3(p,Q)}{\partial Q}.$$

c) If the Hamiltonian is $H(q,p) = p^2 + \alpha^2 q^2$, find the new Hamiltonian generated by the transformation $K(Q,P)$.

d) Find the infinitesimal contact transformation generated by the $F_2(q,P)$ transformation

$$F_2(q,P) = \sum_i q_i P_i + \varepsilon \sum_i \left(x_i p_{yi} - y_i p_{xi}\right)$$

Consider only the coordinate components.

e) What is the physical interpretation of this infinitesimal contact transformation.

QUESTION 2. (7.5 Marks)

A heavy particle slides without friction on circular hoop of radius a, under the influence of gravity. Use the method of Lagrange's undetermined multipliers to **obtain the equations of motion** for the system in polar coordinates (r,θ).
The Euler-Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk} = Q_k \qquad k = 1,2,\ldots,n.$$

Show that the equations of motion are

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + mg\cos\theta = \lambda$$
$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mgr\sin\theta = 0$$

Note that the kinetic energy in polar coordinates is

$$T = \tfrac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right).$$

Using the initial condition $\dot{\theta} = 0$ at $\theta = 0$, show that the Lagrange multiplier is given by

$$\lambda = mg(3\cos\theta - 2).$$

When does the particle separate from the hoop?