THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

NOVEMBER 2008

PHYS3510

Advanced Mechanics, Fields and Chaos

Time Allowed – 2 hours

Total number of questions - 4

Answer ALL questions

All questions ARE of equal value

Candidates may not bring their own calculators.

The following materials will be provided by the Enrolment and Assesment Section: Calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

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FORMULA SHEET

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k}=\sum_l^m \lambda_l a_{lk}=Q_k$$

Canonical Transformations

1)	$F_1(q,Q,t)$	$p_i=\frac{\partial F_1}{\partial q_i}$	$P_i=-\frac{\partial F_1}{\partial Q_i}$	$K=H+\frac{\partial F_1}{\partial t}$
2)	$F_2(q,P,t)$	$p_i=\frac{\partial F_2}{\partial q_i}$	$Q_i=\frac{\partial F_2}{\partial P_i}$	$K=H+\frac{\partial F_2}{\partial t}$
3)	$F_3(p,Q,t)$	$q_i=-\frac{\partial F_3}{\partial p_i}$	$P_i=-\frac{\partial F_3}{\partial Q_i}$	$K=H+\frac{\partial F_3}{\partial t}$
4)	$F_4(p,P,t)$	$q_i=-\frac{\partial F_4}{\partial p_i}$	$Q_i=\frac{\partial F_4}{\partial P_i}$	$K=H+\frac{\partial F_4}{\partial t}$

Poisson Bracket $\quad [u,v]_{q,p}=\sum_k\left(\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k}-\frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}\right)$

Hamilton-Jacobi Theory	$H(q,p,t)$	$H(q,p)=$ constant
Canonical transformation to	$Q_i,\ P_i$	P_i (constant of motion)
New Hamiltonian	$K=0$	$K=H(P_i)=\alpha_1$
New equations of motion	$\dot{Q}_i=\frac{\partial K}{\partial P_i}=0$ $\dot{P}_i=-\frac{\partial K}{\partial Q_i}=0$	$\dot{Q}_i=\frac{\partial K}{\partial P_i}=\nu_i$ $\dot{P}_i=-\frac{\partial K}{\partial Q_i}=0$
With solutions Generating Function	$Q_i=\beta_i,\quad P_i=\gamma_i$ Hamilton's Principle $S(q,P,t)$	$Q_i=\nu_i t+\beta_i,\quad P_i=\gamma_i$ Hamilton's Characteristic $W(q,P)$
Hamilton-Jacobi equation	$H\left(q_i,\frac{\partial S}{\partial q_i},t\right)+\frac{\partial S}{\partial t}=0$	$H\left(q_i,\frac{\partial W}{\partial q_i}\right)-\alpha_1=0$
New constant momenta (one choice)	$P_i=\gamma_i(\alpha_1,\ldots,\alpha_n)$ $\gamma_i=\alpha_i$	$P_i=\gamma_i(\alpha_1,\ldots,\alpha_n)$ $\gamma_i=\alpha_i$

Hamilton-Jacobi solution	$S = S(q_i, \gamma_i, t)$	$W = W(q_i, \gamma_i)$
First half of transformation	$p_i = \dfrac{\partial S}{\partial q_i}$	$p_i = \dfrac{\partial W}{\partial q_i}$
Second half of transformation	$Q_i = \dfrac{\partial S}{\partial \gamma_i} = \beta_i$	$Q_i = \dfrac{\partial W}{\partial \gamma_i} = \nu_i(\gamma_j)t + \beta_i$

Action-angle Variables $\quad J_i = \oint p_i dq_i = \oint \frac{\partial W_i(q_i, \alpha_1, \ldots, \alpha_n)}{\partial q_i} dq_i \qquad w_i = \frac{\partial W}{\partial J_i}$

Mathematical identities

$$\sin^2 Q + \cos^2 Q = 1 \qquad \tan^2 Q + 1 = \sec^2 Q$$

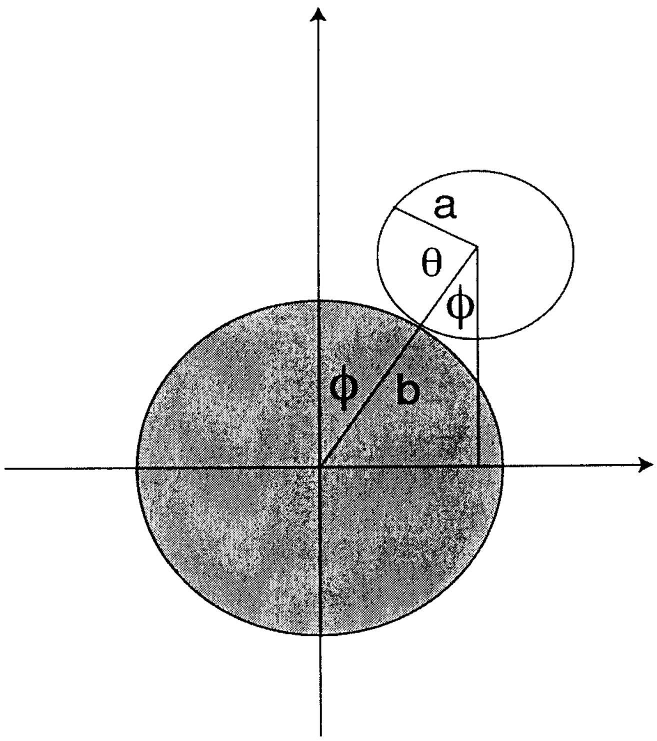
$$1 + \cos 2\phi = 2\cos^2 \phi \qquad 1 - \cos 2\phi = 2\sin^2 \phi$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = \operatorname{cosec}^2 x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

QUESTION 1. (20 marks)



A sphere of radius a and mass m, with moment of inertia $I = \frac{2}{5}ma^2$, rests on top of a fixed sphere of radius b. The first sphere is slightly displaced so that it rolls without slipping down the second sphere.

(a) Show that the Lagrangian is

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + x^2\dot{\phi}^2\right) + \tfrac{1}{5}ma^2\left(\dot{\theta} + \dot{\phi}\right)^2 - mgx\cos\phi$$

where x is the distance between the centres of the two spheres.

(b) Using Lagrange's equations for a constrained system;

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_l^m \lambda_l a_{lk} \qquad k = 1,2,\ldots,n$$

Find the angle ϕ at which the first sphere is no longer in contact with the second sphere.

QUESTION 2. (20 marks)

a) Under what conditions is the transformation

$$Q = q^{\alpha} \cos \beta p, \qquad P = q^{\alpha} \sin \beta p$$

canonical?

b) If the Hamiltonian in the new variables is $K = Q^2 + P^2$, find the Hamiltonian for the old variables and find the time evolution of the old variables q and p.

c) If the generating function $F_2(q,P) = \sum q_i P_i + \varepsilon G(q,P)$ generates and infinitesimal contact transformation, find the change in q and p.

d) If $G \equiv H(q,p)$ where H is the Hamiltonian, determine the contact transformation generated by H.

QUESTION 3.   (20 marks)

Find the frequencies of a two-dimensional simple harmonic oscillator with mass m and unequal force constants k_1 and k_2 in the x and y directions respectively, using the method of action-angle variables. The Hamiltonian is given by

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}\left(k_1 x^2 + k_2 y^2\right).$$

QUESTION 4. (20 marks)

a) Determine the stability properties of equations

$$\dot{x} = x + y - x\left(x^2 + y^2\right)$$
$$\dot{y} = -x + y - y\left(x^2 + y^2\right).$$

b) The stability of an iterative mapping $x_{n+1} = f(x_n)$ can be determined by calculating the Lyapunov exponent defined by

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln\left|f'(x_i)\right|$$

Find the Lyapunov exponent as a function of μ for the two fixed points of the quadratic map, $x_{n+1} = \mu x_n(1 - x_n)$. What is the stability of the fixed points at $\mu = 2$?

c) What are tangent bifurcations and pitchfork bifurcations and how do they arise?

d) For the quadratic map we can describe the type of bifurcations that lead to cycles of a particular length. Complete all the missing entries in the table.

cycle length	periodic points	'prime' points	number of n-cycles	created by	created by
n	2^n			tangent	pitchfork
1	2	2	2	1	0
2	4	2	1	0	1
3	8	6	2	1	0
4	16				
5	32				
6	64				
7	128				
8	256				
12	4096				

e) Define the unstable manifold of a fixed point. Prove that unstable manifolds from different fixed points do not intersect.