University of New South Wales – School of Physics

PHYS 3011 / PHYS 3230 – Electrodynamics
Mid-term Test – Thursday, 18 April 2013
Time: 50 minutes

Please PRINT your name and number.
Write your answers in ink. Please do not use red ink.
Start each answer on a new page. Show all working.

There are **three** questions on this paper: answer **all** questions (total 50 marks)

Q.1 (15 marks)
An electric field is given by: $\mathbf{E} = (2xy + x^3)\,\hat{\mathbf{x}} + (x^2 + 2yz^2)\,\hat{\mathbf{y}} + 2y^2z\,\hat{\mathbf{z}}$.
Show that this field is conservative, and find an expression for the electric potential, $V(x, y, z)$

Q.2 (15 marks)
If $\mathbf{B}$ is uniform, show that $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$, where $\mathbf{r}$ is the vector from the origin to the point in question. That is, show that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$.

Hint: You may want to remind yourself of the results for $\nabla \cdot \mathbf{r}$ and $\nabla \times \mathbf{r}$ first.

Q.3 (20 marks)
A laser beam has a power of 30GW and a diameter of 3mm. Calculate the peak value of E. What is the electric field strength of the laser inside glass of refractive index 1.5? (Assume $\mu_r = 1$ inside the medium.)

OUTLINE ANSWERS

(1) $\nabla \times \mathbf{E} = \left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y},\ \ \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z},\ \ \frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$

$$= (4yz - 4yz,\ \ 0 - 0,\ \ 2x - 2x)\ =\ (0,\ 0,\ 0)$$

So the field is conservative.

$$V = -\int \mathbf{E} \cdot \mathbf{d}\boldsymbol{\ell}$$

$$= -\int_0^x (2xy + x^3)\,dx\Big|_{y=z=0} - \int_0^y (x^2 + 2yz^2)\,dy\Big|_{x=x,\ z=0} - \int_0^z 2y^2z\,dz\Big|_{x=x,\ y=y}$$

$$= -\frac{x^4}{4} - x^2y - y^2z^2 + C$$

(2) Reminder of $\nabla \times \mathbf{r}$ and $\nabla \cdot \mathbf{r}$:

$$\nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \qquad \nabla \cdot \mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

If $\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$ ($\mathbf{B}$ is constant)

Then $\nabla \cdot \mathbf{A} = -\frac{1}{2}\left\{\mathbf{B} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \mathbf{B})\right\} = 0 + 0 = 0$

and $\nabla \times \mathbf{A} = -\frac{1}{2}\left\{(\mathbf{B} \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{r}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{r})\right\}$

$$= -\tfrac{1}{2}\left\{(B_x \frac{\partial}{\partial x})\mathbf{r} + (B_y \frac{\partial}{\partial y})\mathbf{r} + (B_z \frac{\partial}{\partial z})\mathbf{r} - 0 + 0 - \mathbf{B} \cdot 3\right\}$$

$$= -\tfrac{1}{2}\left\{B_x\,\hat{\mathbf{i}} + B_y\,\hat{\mathbf{j}} + B_z\,\hat{\mathbf{k}} - 3\mathbf{B}\right\} = -\tfrac{1}{2}\left\{\mathbf{B} - 3\mathbf{B}\right\} = \mathbf{B}$$

P.T.O.

(3)(a) $N = \dfrac{3 \times 10^{10}}{\pi \times (1.5 \times 10^{-3})^2} = 4.244 \times 10^{15}\ \mathrm{W.m^{-2}}$

$$N = \frac{E_0^2}{Z_0} \qquad \therefore\ E_0 = \sqrt{N Z_0} = \sqrt{4.244 \times 10^{15} \times 376.7}$$
$$= 1.264 \times 10^9 = 1.26\,\mathrm{GV/m} \quad (\text{This is } E_{\mathrm{rms}})$$

So $E_{\mathrm{peak}} = \sqrt{2} \times 1.264 \times 10^9 = 1.79\,\mathrm{GV/m}$

(b) $$Z = \frac{Z_0}{n} \qquad \therefore\ E = \sqrt{N Z} = \sqrt{\frac{N Z_0}{n}} = \frac{E_0}{\sqrt{n}} = \frac{1.264 \times 10^9}{\sqrt{1.5}}$$
$$= 1.032 \times 10^9 = 1.03\,\mathrm{GV/m}\ (\mathrm{rms})$$

So $E_{\mathrm{peak}} = \sqrt{2} \times 1.032 \times 10^9 = 1.46\,\mathrm{GV/m}$

Alternatively:

$u_0 = \epsilon_0 E_0^2 \qquad u = \epsilon_r \epsilon_0 E^2 \qquad$ but $uv = u_0 c \quad \therefore\ u = u_0 c/v = u_0 n$

$$\therefore\ \epsilon_r \cancel{\epsilon_0} E^2 = \cancel{\epsilon_0} E_0^2\, n = E_0^2 \sqrt{\epsilon_0}$$
$$\therefore\ E^2 = E_0^2/\sqrt{\epsilon_r} = E_0^2/n$$
$$\therefore\ E = E_0/\sqrt{n} = 1.03\,\mathrm{GV/m}\ (\mathrm{rms}),\ \ 1.46\,\mathrm{GV/m}\ (\mathrm{peak}).$$

Useful Formulae: PHYS3011/PHYS3230

$\epsilon_0 = 8.854 \text{ x } 10^{-12}\ \mathrm{Fm^{-1}} \quad \mu_0 = 4\pi \times 10^{-7}\ \mathrm{Hm^{-1}} \quad c = 3 \times 10^8\ \mathrm{ms^{-1}}$

Volume element $= dx\,dy\,dz \;=\; r^2 \sin\theta\, dr\, d\theta\, d\phi$

Surface area of sphere $= 4\pi r^2$ Volume of sphere $= \frac{4}{3}\pi r^3$

Divergence Theorem: $\int_V \nabla\cdot\mathbf{A}\, dV = \int_S \mathbf{A}\cdot\mathbf{dS}$ (S is the surface enclosing V)

Stokes' Theorem: $\int_S (\nabla\times\mathbf{A})\cdot\mathbf{dS} = \oint_L \mathbf{A}\cdot\mathbf{dl}$ (L is the curve bounding S)

Vector identity: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$

So: $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\,\mathbf{E}$

Also: $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$

Dielectric materials:

$\mathbf{P} = \chi\epsilon_0\mathbf{E} \qquad \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = (1+\chi)\epsilon_0\mathbf{E} = \epsilon_r\epsilon_0\mathbf{E}$

E field (and potential difference) is reduced by a factor ϵ_r in the bulk.

Energy density, $u = \frac{1}{2}\epsilon_r\epsilon_0 E^2$ per unit volume $= \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$

Gauss's Law for **D** : $\int \mathbf{D}\cdot\mathbf{dS} = q_{free} \qquad \nabla\cdot\mathbf{D} = \rho_{free}$

At a boundary, $E_\parallel$ and V are continuous ($D_\perp$ is continuous.)

Cavities in dielectrics: $\mathbf{E}_{\mathrm{local}} = \mathbf{E}_{\mathrm{bulk}}$ for a needle-shaped cavity;

$\mathbf{E}_{\mathrm{local}} = \mathbf{E}_{\mathrm{bulk}} + \mathbf{P}/\epsilon_0$ for a disc-shaped cavity;

$\mathbf{E}_{\mathrm{local}} = \mathbf{E}_{\mathrm{bulk}} + \mathbf{P}/3\epsilon_0$ for a spherical cavity.

Clausius-Mossotti equation: $\dfrac{n\alpha}{3\epsilon_0} = \left(\dfrac{\epsilon_r - 1}{\epsilon_r + 2}\right)$

Capacitance:

stored charge $Q = C\Delta V$ [C]
stored energy $U = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2$ or $\frac{1}{2}Q^2/C$ [J]

capacitance of parallel-plate capacitor is $C = \epsilon_r\epsilon_0 A/d$ [F]

capacitance of isolated sphere is $C = 4\pi\epsilon_r\epsilon_0 R$ [F]

DC Circuits:

Ohm's Law: $\Delta V = IR$ resistance, $R = \rho l/A$ [Ω]

Kirchhoff's Laws: (1) $\Sigma I = 0$ at a junction
(2) $\Sigma \mathcal{E} - \Sigma IR = 0$ around each loop

Joule heating: power dissipated, $P = I\Delta V = I^2R = (\Delta V)^2/R$ [W]

Ohm's law: $\mathbf{J} = \sigma\mathbf{E}$ power dissipated/unit volume $= \mathbf{J}\cdot\mathbf{E} = \sigma E^2$

Magnetic media:

$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu_r\mu_0\mathbf{H}$ *ie* $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$

$\nabla\cdot\mathbf{B} = 0$, so $\nabla\cdot\mathbf{H} + \nabla\cdot\mathbf{M} = 0$

Ampère's law becomes: $\nabla\times\mathbf{H} = \mathbf{J}_{free}$

At a boundary, $B'_\perp = B_\perp$ and $H'_\| = H_\|$

Inductance:

Mutual inductance: $\Phi_1 = L_{12}I_2$, $\Phi_2 = L_{12}I_1$, Self inductance: $\Phi = LI$

Self Inductance of a solenoid: $L = \mu_r\mu_0\dfrac{N^2}{\ell}A$ magnetic energy: $U = \frac{1}{2}LI^2$

Energy density in magnetic field: $u = \frac{1}{2}\dfrac{B^2}{\mu_r\mu_0} = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$

Maxwell's Equations

In a vacuum:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Lorentz force law: $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

Maxwell's equations in dielectric and magnetic media:

$$\nabla\cdot\mathbf{D} = \rho \qquad \nabla\times\mathbf{E} = -\dot{\mathbf{B}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}$$

EM Waves:

Wave equation for $\mathbf{E}$ in free space: $\nabla^2\mathbf{E} = \mu_0\epsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ *ie* $c = 1/\sqrt{\mu_0\epsilon_0}$

(in a medium: $v = 1/\sqrt{\mu_r\mu_0\epsilon_r\epsilon_0} = c/n$, n = refractive index)

Solution: $E_x = E_0\sin(kx - \omega t)$ for monochromatic wave travelling in +ve x-direction.

$\mathbf{E}$, $\mathbf{B}$ and the direction of propagation $\hat{\mathbf{k}}$ are mutually perpendicular:

$\hat{\mathbf{k}}\cdot\mathbf{E} = 0$ $\hat{\mathbf{k}}\cdot\mathbf{B} = 0$ $c\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}$ $\hat{\mathbf{E}}\times\hat{\mathbf{B}} = \hat{\mathbf{k}}$

The direction of $\mathbf{E}$ is the direction of polarization of the E-M wave.

Impedance of free space, $Z_0 = \dfrac{|E|}{|H|} = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = 377\,\Omega$

Poynting vector: $\mathbf{N} = \mathbf{E}\times\mathbf{H} = \dfrac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = E^2/Z_0 = H^2Z_0$

NB: Wave number, $k = \dfrac{2\pi}{\lambda}$ Angular frequency, $\omega = 2\pi f$

Phase velocity, $v = f\lambda = \dfrac{\omega}{k}$ Group velocity $= \dfrac{d\omega}{dk}$

NB There will be more formulae in the final exam !