Phys 3230 Electromagnetism

Answers to questions at mid-session Exam 2005

1. Fields, potentials, gauge invariance and Maxwell's equations

i. Write down expressions for the electric **E** and magnetic **B** fields via the scalar and vector potentials V and **A**.

$$\mathbf{E} = -\nabla V - \dot{\mathbf{A}}, \quad \mathbf{B} = \nabla \times \mathbf{A} \qquad (1.1)$$

ii. Write down the second pair of the Maxwell's equations (equations, which do not include charges and currents).

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\dot{\mathbf{B}} \qquad (1.2)$$

iii. Prove that expressing the fields via the potentials one automatically satisfies the second pair of the Maxwell's equations.

$$\begin{aligned} &\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0, \\ &\nabla \times \mathbf{E} = \nabla \times \left(-\nabla V - \dot{\mathbf{A}}\right) = \\ &-\nabla \times \dot{\mathbf{A}} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A} = -\dot{\mathbf{B}} \end{aligned} \qquad (1.3)$$

iv. Write down gauge transformations for the potentials. Verify that under these transformations the fields do not change.

$$\begin{aligned} &V \to V' = V - \dot{\alpha} \\ &\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \alpha \\ &\mathbf{E} \to \mathbf{E}' = \mathbf{E} - \nabla\left(-\dot{\alpha}\right) - \frac{\partial}{\partial t} \nabla \alpha = \mathbf{E} \\ &\mathbf{B} \to \mathbf{B}' = \mathbf{B} + \nabla \times \left(\nabla \alpha\right) = \mathbf{B} \end{aligned} \qquad (1.4)$$

2. Conservation laws

i. Describe briefly the physical meaning of $u_{em} = \frac{\varepsilon_0 \mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0}$ and $\mathbf{S} = \frac{\mathbf{E} \times \mathbf{B}}{\mu_0}$.

- u_{em} is the energy density of the electromagnetic field.
- **S** is the Poynting vector, which describes the energy flux; $\mathbf{S}/c^2$ gives the momentum density of the electromagnetic field.

ii. Write down the energy conservation law for the system, which consists of the electromagnetic field and charged particles.

$$\frac{\partial}{\partial t}\left(u_{mech} + u_{em}\right) + \nabla \cdot \mathbf{S} = 0 \qquad (2.1)$$

Here u_{mech} is the density of energy of particles.

iii. Describe briefly the physical meaning of

$$\overleftrightarrow{T}_{ij} = \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2 \right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}\mathbf{B}^2 \right) \quad (2.2)$$

- Eq.(2.2) defines the Maxwell stress tensor.
- $\overleftrightarrow{T}_{ij} ds_j$ gives the force in the direction *i*, which is applied to the element ds_j of the surface that surrounds the given volume; $\oint \overleftrightarrow{T} \cdot d\vec{s}$ equals the total force, which is applied to all particles in the given volume.
- Alternatively, $\left(-\overleftrightarrow{T}_{ij}\right)$ describes the flux of the momentum of the electromagnetic field, see Eq.(2.3).

iv. Write down the momentum conservation law for the system, which consists of the electromagnetic field and charged particles.

$$\frac{\partial}{\partial t}\left(\vec{P}_{mech} + \vec{P}_{em}\right) + \nabla \cdot \left(-\overleftrightarrow{T}\right) = 0,$$

$$\vec{P}_{em} = \frac{1}{c^2}\mathbf{S} = \varepsilon_0 \mathbf{E} \times \mathbf{B}, \quad (2.3)$$

$\vec{P}_{mech}$ is the density of momentum of particles

3. Applications

i. Consider static homogeneous magnetic field **B**=(0,0,B). Write down the stress tensor, presenting it as a 3 by 3 matrix.
Eq.(2.2) gives

$$\overleftrightarrow{T} = \frac{B^2}{2\mu_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.1)$$

ii. Suppose that within some finite volume there exist static homogeneous fields **E** and **B.** Assume that the total energy W, which is accumulated in these fields is fixed, W= *const*.
Find the fields **E**, **B**, which provide the maximum value for the momentum **|P|** of the electromagnetic field. Answering this question

- Describe a relative orientation of these fields **E** and **B**.

$$|\mathbf{P}| = \varepsilon_0 |\mathbf{E} \times \mathbf{B}| V = \varepsilon_0 EB \sin\theta V \quad (3.2)$$

Here θ is the angle between **E** and **B,** V is the volume. To make | **P** | large one needs to choose

$$\sin\theta = 1 \quad (3.3)$$

This means that **E** and **B** are orthogonal

$$\mathbf{E} \perp \mathbf{B} \quad (3.4)$$

From Eqs.(3.2),(3.3) one finds

$$|\mathbf{P}| = \varepsilon_0 EBV \quad (3.5)$$

which will be helpful below.

- Find the ratio |**E**| / |**B**|.

  Remember that

$$xy \le \frac{x^2 + y^2}{2} \qquad (3.6)$$

  The identity in Eq. (3.6) is achieved only when

$$x = y \qquad (3.7)$$

  Eq.(3.5) can be looked at as the left-hand side of Eq.(3.6)

$$c\,|\mathbf{P}| = xy, \quad x = \sqrt{\varepsilon_0 V}\,E, \quad y = \sqrt{\frac{V}{\mu_0}}\,B \qquad (3.8)$$

  where $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$ is the velocity of light

  The energy of the field W can be written as the right-hand side of Eq.(3.6)

$$\mathrm{W} = u_{em} V = \left(\frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) V = \frac{x^2 + y^2}{2}$$

  (3.9)

  From Eqs.(3.6)-(3.9) one concludes that

$$c\,|\mathbf{P}| \le \mathrm{W} \qquad (3.10)$$

  An equality in Eq.(3.10) needs that $x = y$, see Eq.(3.7). From here, using Eq.(3.8) one finds

$$E = cB$$
$$\frac{|\mathbf{E}|}{|\mathbf{B}|} = c \qquad (3.11)$$

- Find the maximum value for the momentum $|\mathbf{P}|_{max}$ of the electromagnetic field.

  From Eq.(3.10) one derives

$$|\mathbf{P}|_{\max} = \frac{W}{c} \qquad (3.12)$$

Eqs.(3.4),(3.11) and (3.12) present an answer to ii.

Arguably, they could have been written without any calculations, from pure *common sense*.