Electromagnetism Phys 3230

Exam 2005

All four questions in Phys3230 should be addressed. If one is not certain in maths, one should try to present explanations in words.

1. **Maxwell's equations (**25% from 100 given for 3230 (non-advanced))
   a. Write down Maxwell's equations in the vacuum.
   b. Write down the current conservation law.
   c. Verify that the current conservation can be derived from the Maxwell's equations.
   d. Write down the momentum conservation law for the electromagnetic field. Explain *very* briefly (not more than a couple of phrases) the physical meaning of
   $$u_{em} = \frac{\varepsilon_0 \mathbf{E}^2}{2} + \frac{\mathbf{B}^2}{2\mu_0} \quad \text{and} \quad \mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}$$
   e. Write down Maxwell equation in matter, assuming that the matter is characterized by the dielectric constant ε and the magnetic constant μ.

2. **Electromagnetic waves (25%)**
   a. Derive the wave equation for the electric field $\mathbf{E}$ in the vacuum, and by analogy, state the wave equation for the a magnetic field **B** in the vacuum.
   b. Write down the wave equations for the fields $\mathbf{E}$ and $\mathbf{B}$ in matter, which is characterized by the dielectric constant ε and the magnetic constant μ.
   c. How does the velocity of light in matter depend on ε and μ ?
   d. What happens with the velocity of light if ε or μ become very large, $\varepsilon, \mu \to \infty$?
   e. Describe very briefly under which conditions $\varepsilon(\omega)$ becomes large.

3. Radiation phenomena (25%)

a. A charged particle is oscillating with the frequency ω and the amplitude *a*, according to $x = a\sin\omega t$. The charge of the particle is *q*.

   i. Assume that $a \ll 2\pi c/\omega$, where *c* is the velocity of light. What type of electromagnetic radiation produces this particle ?

   ii. Write down the energy loss $\frac{dW}{dt}$ (energy per second) due to the electromagnetic radiation produced by this particle.

Keep in mind that the radiation is produced by the dipole moment of the particle. (Indicate which condition makes the dipole E1 radiation dominant.)

Hint: you do not necessarily need to remember by heart the necessary formulas. Just remember that the dipole radiation depends on only two parameters related to the particle, its maximum dipole moment $d_{\max}$ and the frequency ω of oscillations. (There are also the velocity of light *c* and ε_0.) Simple dimensional counting shows that there is only one combination of these parameters that has the necessary dimension to reproduce $\frac{dW}{dt}$. The coefficient, which cannot be reproduced by this simple method, is not important here.

b. There is a current over a circled wire. The current oscillates $I(t) = I_{\max}\sin\omega t$, the wire has a surface area $S = \pi a^2$.

   i. What condition on ω, a should be satisfied in order that this device produces mainly M1 radiation ?

   ii. Write down the energy loss $\frac{dW}{dt}$ due to the radiation.

4. Electromagnetic field in matter (25%)

Consider a dipole moment $\mathbf{m}$ located on the z-axis at the point $z = z_0 > 0$. Assume that the dipole is oriented along the z-axis $\mathbf{m} = (0, 0, m)$ and that the plane xy represents a boundary of a superconductor, which fills in the volume $z \leq 0$.

a. Find the magnetic field, which this device (the dipole + superconductor) produces in the vacuum.
b. Find the absolute value and direction of the force between the magnetic moment and the superconductor.
c. Outline *very* briefly which practical applications (if any) this device could have.

Hint. Keep in mind that the superconductor reacts on an external magnetic field in such a way as to keep the magnetic field inside zero. As a result, the normal component of the total magnetic field (dipole + superconductor) must be absent on the surface, i.e.

$$B_z = 0 \text{ when } z = 0 \tag{2.1}$$

The handy trick is that the condition Eq.(2.1) can be satisfied if the magnetic field produced by the superconductor is replaced by the field produced by some "additional" magnetic moment.

- Find from Eq.(2.1) a location of this "additional" moment, its magnitude, and its orientation.

Solving this problem it could be handy to remember that the magnetic field of the dipole in the vacuum reads

$$\mathbf{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}, \quad \mathbf{n} = \frac{\mathbf{r}}{r} \tag{2.2}$$

where $\mathbf{r}$ is the radius-vector from the dipole to the observation point.

Answers to questions in Part I

Please note that these answers are presented in a brief, simplified notation (and missprints are possible).

I. **Maxwell's equations**

1)
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

2)
$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{2.4}$$

3)
$$\nabla \cdot \dot{\mathbf{E}} = \frac{\dot{\rho}}{\varepsilon_0}$$
$$\nabla \cdot (\nabla \times \mathbf{B}) = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla \cdot \mathbf{E} + \mu_0 \nabla \cdot \mathbf{J} = 0$$
$$c^2 = \frac{1}{\varepsilon_0 \mu_0} \tag{2.5}$$
$$\varepsilon_0 \mu_0 \nabla \cdot \dot{\mathbf{E}} + \mu_0 \nabla \cdot \mathbf{J} = 0$$

4)
$$\dot{\rho} + \nabla \cdot \mathbf{J} = 0$$
$$\dot{u}_{em} + \nabla \cdot \mathbf{S} = 0 \tag{2.6}$$

Here u_{em} and $\mathbf{S}$ are the density of energy and the Poynting vector, the latter represents the flux of energy.

5)
$$\nabla \cdot \mathbf{D} = \rho_{ext} \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.7}$$
$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon \varepsilon_0 \mathbf{E}$$
$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} = \mu \mu_0 \mathbf{H}$$

II. **Electromagnetic waves**

1)
$$\nabla\cdot\mathbf{E}=0 \qquad \nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\times(\nabla\times\mathbf{B})=\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\mathbf{E}$$
$$\nabla(\nabla\cdot\mathbf{B})-\Delta\mathbf{B}=-\frac{1}{c^2}\frac{\partial}{\partial t}\frac{\partial}{\partial t}\mathbf{B} \tag{3.1}$$
$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}=0$$
$$\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}=0$$

2)
$$\left(\Delta-\frac{1}{\mathrm{v}^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}=0,\quad \left(\Delta-\frac{1}{\mathrm{v}^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}=0 \tag{3.2}$$

3)
$$\mathrm{v}=\frac{c}{\sqrt{\varepsilon\mu}} \tag{3.3}$$

4)
$$\mathrm{v}\to 0 \quad \text{when } \varepsilon \text{ or } \mu\to\infty \tag{3.4}$$

5) A vicinity of some resonance.

III. Radiation phenomena

1) Charged particle:

There is only a charge, no current.
Therefore the radiation is electric.

i. $a \ll \lambda=\frac{2\pi\,\omega}{c}$ means that the higher harmonics are suppressed by a factor $(a/\lambda)^{2l}$, where $l=1,2...$ is the multipole moment. The dipole radiation $l=1$ dominates. Thus, it is the electric dipole radiation, called E1. (3.5)

ii.
$$\frac{dW}{dt}=\frac{1}{3}\left(\frac{1}{4\pi\varepsilon_0}\right)\left(\frac{d_{\max}^2\omega^4}{c^3}\right),\quad d_{\max}=qa \tag{3.6}$$

2) Current:

i.
$$a\ll\lambda=\frac{2\pi\omega}{c} \tag{3.7}$$

ii.
$$\frac{dW}{dt}=\frac{1}{3}\left(\frac{\mu_0}{4\pi}\right)\left(\frac{m_{\max}^2\omega^4}{c^3}\right),\quad m_{\max}=I_{\max}S \tag{3.8}$$

IV. **Superconductor**:

The dipole moment located at z_0 and oriented along the $z-$axis has the components

$$\mathbf{m} = (0, 0, m)$$

In order to eliminate the normal component of the magnetic field created by this dipole on the surface one needs to assume that there exists the "mirror" dipole moment located inside the superconductor at $z = -z_0$, and oriented in the direction opposite to the original one

$$\mathbf{m}' = -\mathbf{m} = -(0, 0, m)$$

The total magnetic field outside the superconductor is described by the combination of the field

$$\mathbf{B} = \left(\frac{\mu_0}{4\pi}\right)\frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3}$$

created by the initial dipole moment ($\mathbf{r}$ is the radius vector from the dipole to the observation point) and the field

$$\mathbf{B}' = \left(\frac{\mu_0}{4\pi}\right)\frac{3(\mathbf{m}'\cdot\mathbf{n}')\mathbf{n}' - \mathbf{m}'}{r'^3} = -\left(\frac{\mu_0}{4\pi}\right)\frac{3(\mathbf{m}\cdot\mathbf{n}')\mathbf{n}' - \mathbf{m}}{r'^3},$$

which is created by the superconductor ($\mathbf{r}'$ is the radius vector from the "mirror" dipole moment to the observation point, $\mathbf{n}' = \mathbf{r}'/r'$). Thus,

$$\mathbf{B}_{tot} = \mathbf{B} + \mathbf{B}' = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{3(\mathbf{m}\cdot\mathbf{n})\mathbf{n} - \mathbf{m}}{r^3} - \frac{3(\mathbf{m}\cdot\mathbf{n}')\mathbf{n}' - \mathbf{m}}{r'^3}\right)$$

The potential energy U of the original dipole in the magnetic field created by the superconductor can be found as its energy in the magnetic field of the "mirror" dipole

$$U = -\mathbf{m}\cdot\mathbf{B}' = -\mathbf{m}\cdot\left[(-)\left(\frac{\mu_0}{4\pi}\right)\frac{3(\mathbf{m}\cdot\mathbf{n}')\mathbf{n}' - \mathbf{m}}{r'^3}\right]_{\mathbf{n}'=(0,0,-1),\,r'=2z_0}$$

$$= \left(\frac{\mu_0}{4\pi}\right)\frac{(3-1)m^2}{r'^3} = \left(\frac{\mu_0}{4\pi}\right)\frac{2m^2}{(2z_0)^3} = \left(\frac{\mu_0}{4\pi}\right)\frac{m^2}{4z_0^3}$$

This potential energy creates the force F between the dipole and the superconductor

$$F = -\frac{\partial U}{\partial z_0} = \left(\frac{\mu_0}{4\pi}\right)\frac{3m^2}{4\,z_0^4}$$

Importantly, $F > 0$, which means **repulsion**. One can fly over the surface of a superconductor. (This effect is in constrast to the attraction, which exists between a charge and a metal surface.)

The following discussion iams at finding the surface current.
This question was not suggested in the exam. Therefore, it is *not* related to the marking of the examination papers.

The tangent component of the magnetic field on the surface reads

$$\mathbf{B}_{\|,total} = 2\mathbf{B}_{\|} = \left(\frac{\mu_0}{4\pi}\right)\frac{6(\mathbf{m}\cdot\mathbf{n})\mathbf{n}_{\|}}{r^3}$$

In the cilindrical coordinates (z is the cilyndrical axes)

$$\mathbf{B}_{\|,total} \to B_{\rho,total} = -6\left(\frac{\mu_0}{4\pi}\right)\frac{m\sin\theta\cos\theta}{r^3} = -6\left(\frac{\mu_0}{4\pi}\right)\frac{m\sin\theta\cos^4\theta}{z_0^3},$$

where θ $(0 \le \theta \le \pi/2)$ is the angle between $(-\mathbf{m})$ and the direction from the dipole to the point on the surface, which implies $\mathbf{m}\cdot\mathbf{n} = -m\cos\theta$, $\mathbf{n}_{\|} = \mathbf{n}_{\rho}$, $|\mathbf{n}_{\|}| = \sin\theta$ and $r = z_0/\cos\theta$.

The boundary condition reads

$$\mathbf{H}_{\|,1} - \mathbf{H}_{\|,2} = \mathbf{K}_f \times \mathbf{v}$$

where $\mathbf{v}$ is the normal to the surface, $\mathbf{H}_{\|,1} = \mathbf{B}_{\|}/\mu_0$ is the field outside, $\mathbf{H}_{\|,2} = 0$ is the field inside. From here, one finds the surface current $\mathbf{K}_f$

$$\mathbf{K}_f = \mathbf{K}_{\phi,f}, \quad K_{\phi,f} = -\left(\frac{1}{4\pi}\right)\frac{6m}{z_0^3}\sin\theta\cos^4\theta$$