THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2010

PHYS3210

Quantum Mechanics

Time Allowed – 2 hours

Total number of questions - 4

Answer ALL questions

All questions are of equal value

Candidates must supply their own, university approved, calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Candidates may keep this paper.

Examination 2010
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This paper may be retained by the candidate.

$$\hbar = 1.054 \times 10^{-34} J \ sec = 6.582 \times 10^{-16} eV \ sec$$

$$1eV = 1.602 \times 10^{-19} J \ .$$

Question 1

A particle moves in a 1-dimensional potential

$$V(x) = \frac{1}{2} m\omega^2 x^2 \, .$$

So the Hamiltonian is $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$.

a. **(4marks)** Show that the functions

$$\psi_0(x) = \alpha_0 e^{-\beta x^2} \, ,$$
$$\psi_1(x) = \alpha_1 x e^{-\beta x^2}$$

are eigenfunctions of the Hamiltonian if

$$\beta = \frac{m\omega}{2\hbar} \, .$$

b. **(2marks)** Determine the values of the energies E_0 and E_1.

c. **(4marks)** Find the ground state energy and the ground state wave function for a particle of mass m moving in the potential

$$V(x) = \begin{cases} \frac{1}{2} m\omega^2 x^2 & at \quad x > 0 \\ \infty & at \quad x < 0 \end{cases}$$

Question 2

The spherical harmonics $|l, m\rangle = Y_{lm}$ are simultaneous eigenfunctions of the operators $\hat{l}^2 = \hat{L}^2/\hbar^2$ and $\hat{l}_z = \hat{L}_z/\hbar$.

a. (1mark) Write down eigenvalue equations for the two operators, showing clearly the possible values of l and m.

The first three rotational energy levels of the PbO molecule are

$$\begin{aligned} \epsilon_0 &= 0, \\ \epsilon_1 &= 0.76 \times 10^{-4} eV, \\ \epsilon_2 &= 2.28 \times 10^{-4} eV. \end{aligned}$$

The molecule consists of isotopes ^{208}Pb and ^{16}O, $M_O = 26.5 \times 10^{-27} kg$, $M_{Pb} = 343 \times 10^{-27} kg$.

b. (2marks) What is the degeneracy of each level? Write the wave functions for all quantum states corresponding to these levels in terms of spherical harmonics Y_{lm}.

c. (3marks) Using the energy levels find the distance between the Lead and the Oxygen nuclei in the PbO molecule.

d. (4marks) Consider the PbO molecule in the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}} Y_{00} + \frac{1}{2} Y_{10} + \frac{1}{2} Y_{21} .$$

This is not a stationary state, but this quantum state can be created by a combination of radio-frequency pulses.

What is the average energy of the state

$$\langle \hat{H} \rangle = ?$$

Present your answer in eV.

Calculate also the following expectation values for this quantum state.

$$\begin{aligned} &\langle l_z \rangle = ? \\ &\langle l_x \rangle = ? \\ &\langle l^2 \rangle = ? \end{aligned}$$

Question 3

The energy of the 2p-state of a hydrogen atom is

$$E = -\frac{1}{8}\frac{me^4}{(4\pi\epsilon_0\hbar)^2} \approx -3.4eV \ .$$

Here m is the electron mass and e is the elementary charge. The electron wave function of the state reads

$$\psi(r) = \frac{1}{2\sqrt{6a_B^3}}\frac{r}{a_B}e^{-r/(2a_B)}\ Y_{1m},$$

where

$$a_B = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 0.529\ 10^{-10}m,$$

is the Bohr radius, and Y_{1m} is the spherical harmonics.

Consider a hydrogen-like uranium ion, i. e. an ion that consists of uranium nucleus and single electron. The charge of the nucleus is $Q = Ze$, where $Z = 92$. Based on Schrodinger equation for electron in the Coulomb field show how the energy of the 2p-state and the wave function of the state scale with Z. Hence calculate

a. (3 marks) The energy of the 2p-state of the uranium ion, present both the expression in terms of fundamental constants and the numerical value of the energy in eV.

b. (3 marks) The electron wave function in the uranium ion 2p-state.

c. (4 marks) The average distance $\langle r\rangle$ between the electron and the uranium nucleus in the 2p-state. The value of the dimensionless integral that you need for the calculation is $\int_0^\infty x^5e^{-x}dx = 5! = 120$.

Question 4

The spin operator $\hat{\vec{s}}$ for a particle with spin 1/2 (for example a neutron) can be expressed in terms of Pauli matrixes $\hat{\vec{\sigma}}$, $\hat{\vec{s}} = \frac{1}{2}\hat{\vec{\sigma}}$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The wave function of a particle polarized along the z-axis is determined by the equation

$$\hat{s}_z \psi_1 = +\frac{1}{2}\psi_1 ,$$

and the wave function of a particle polarized along the x-axis is determined by the equation

$$\hat{s}_x \psi_2 = +\frac{1}{2}\psi_2 .$$

a. (4marks) Find the wave functions ψ_1 and ψ_2.

A neutron is in the state ψ_2. Instantly a magnetic field B directed along the z-axis is switched on. The spin interacts with the magnetic filed, the Hamiltonian is

$$\hat{H} = -\mu B \sigma_z ,$$

where μ is the magnetic moment of neutron.

b. (4marks) Solve the time dependent Schrodinger equation

$$i\hbar \frac{\partial \psi(t)}{\partial t} = \hat{H}\psi(t)$$

$$\psi = \begin{pmatrix} a \\ b \end{pmatrix}$$

and hence find $a(t)$ and $b(t)$.

c. (2marks) Calculate the expectation value

$$\langle \psi(t) | \hat{s}_x | \psi(t) \rangle$$

and hence determine how the spin precesses around the z-axis.