THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

MID-SESSION EXAMINATION

APRIL 2008

PHYS3020

Statistical Physics

Time Allowed – 50 minutes

Total number of questions - 3

Answer ALL questions

All questions ARE of equal value

Candidates may bring their own calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

FORMULA SHEET

Boltzmann Entropy $S = k\ln W$

Statistics and Distributions

Boltzmann $W_B = N!\prod_{j=1}^{n}\frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z}e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Maxwell-Boltzmann $W_{MB} = \prod_{j=1}^{n}\frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z}e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Fermi-Dirac $W_{FD} = \prod_{j=1}^{n}\frac{g_j!}{N_j!\left(g_j - N_j\right)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j-\mu)/kT}+1}$

Bose-Einstein $W_{BE} = \prod_{j=1}^{n}\frac{(N_j + g_j - 1)!}{N_j!\left(g_j - 1\right)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j-\mu)/kT}-1}$

Microcanonical $f_{mc}(q,p) = \frac{\delta\left(H(q,p)-E\right)}{\int dqdp\,\delta\left(H(q,p)-E\right)}$

Canonical $f_C(q,p) = \frac{\exp\left(-\beta H(q,p)\right)}{Z(N,V,T)}$ $Z(N,V,T) = \int dqdp\exp\left(-\beta H(q,p)\right)$

Grand-canonical $f_G(q,p) = \frac{\exp\left(\beta\left(\mu N - H\right)\right)}{\Xi\left(\mu,V,T\right)}$ $\Xi\left(\mu,V,T\right) = \sum_{N=0}^{\infty} z^N Z\left(N,V,T\right)$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2\left(\frac{\partial \ln Z}{\partial T}\right)_V$ Pressure $P = kT\left(\frac{\partial \ln Z}{\partial V}\right)_T$

Mathematical identities

$$\ln N! \approx N\ln N - N \qquad \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \qquad \frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$$

$$1 + y + y^2 + \ldots = \frac{1}{1-y} \qquad \int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha} \qquad \int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh(x) = \cosh(x) \qquad \mathrm{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh(x) = \sinh(x) \qquad \mathrm{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \qquad \frac{d}{dx}\coth(x) = -\mathrm{csch}^2(x)$$

QUESTION 1 (5 marks)

(a) Show that the thermodynamic probability

$$W = \prod_{j=1}^{n} \frac{g_j(g_j - a)(g_j - 2a)...(g_j - (N_j - 1)a)}{N_j!}$$

Reduces to Maxwell-Boltzmann statistics when $a = 0$, to Fermi-Dirac statistics when $a = 1$, and to Bose-Einsein statistics when $a = -1$.

(b) If the density of states $g(\varepsilon)d\varepsilon = \left(4\sqrt{2}\pi V/h^3\right)m^{3/2}\varepsilon^{1/2}d\varepsilon$, approximate the sum in the partition function

$$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$$

and show that

$$Z = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V$$

(c) Calculate the internal energy and pressure for this system using

$$U = NkT^2\left(\frac{\partial \ln Z}{\partial T}\right)_V \quad \text{and} \quad P = NkT\left(\frac{\partial \ln Z}{\partial V}\right)_T .$$

(d) If the Helmholtz function $F = -NkT(\ln(Z/N) + 1)$, show that the entropy of the system is

$$S = Nk\left(\frac{3}{2}\ln T - \ln\left(\frac{N}{V}\right)\right) + S_0$$

and find the value of the constant S_0.

QUESTION 2 (5 marks)

(a) The classical canonical partition function is given by

$$Z(\beta) = \int \exp\left(-\beta H(q,p)\right) dq dp$$

where $H = \frac{1}{2m}\sum \mathbf{p}_i^2 + \Phi(q)$ is the Hamiltonian (or energy) of the system and $dqdp$ denotes the integral over all positions and momenta appearing in the Hamiltonian. For an ideal gas the potential function Φ is zero so the partition function can be evaluated. Show that

$$Z = V^N \left(2\pi mkT\right)^{3N/2}$$

(b) Find the entropy of this system given that

$$S = \frac{\partial}{\partial T}\left(kT \ln Z\right)$$

(c) Why is this entropy different to that obtained using the quantum approach, that is

$$S = Nk\left\{\frac{5}{2} + \ln\left(\frac{V}{N}\right) + \frac{3}{2}\ln\left(\frac{2\pi mkT}{h^2}\right)\right\}$$

(d) How is the difference between the quantum and classical entropy usually resolved?

QUESTION 3 (5 marks)

(a) The classical grand canonical partition function is given by

$$\Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T).$$

where z is the fugacity and $Z(N,V,T)$ is the canonical partition function. Show that the average number of particles $\langle N \rangle$ is given by

$$\langle N \rangle = z\frac{\partial}{\partial z}\ln\Xi \qquad \text{where} \qquad \langle N \rangle \equiv \frac{1}{\Xi}\sum_{N=0}^{\infty} N z^N Z(N,V,T).$$

(b) If $\Delta N = N - \langle N \rangle$, show that $\langle \Delta N^2 \rangle = \langle N^2 \rangle - \langle N \rangle^2$.

(c) Show that

$$\langle N(N-1) \rangle = \frac{1}{\Xi} z^2 \frac{\partial^2}{\partial z^2}\Xi.$$

(d) And therefore show that

$$\langle N^2 \rangle = \frac{1}{\Xi} z^2 \frac{\partial^2}{\partial z^2}\Xi + \frac{1}{\Xi} z \frac{\partial}{\partial z}\Xi$$