THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE/JULY 2007

PHYS3020

Statistical Physics

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

All questions ARE of equal value

Candidates may not bring their own calculators.

The following materials will be provided by the Enrolment and Assesement Section: Calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

FORMULA SHEET

Boltzmann Entropy $S = k\ln W$

Statistics and Distributions

Boltzmann	$W_B = N!\prod_{j=1}^{n}\frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z}e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Maxwell-Boltzmann	$W_{MB} = \prod_{j=1}^{n}\frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z}e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Fermi-Dirac	$W_{FD} = \prod_{j=1}^{n}\frac{g_j!}{N_j!\left(g_j - N_j\right)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j-\mu)/kT}+1}$
Bose-Einstein	$W_{BE} = \prod_{j=1}^{n}\frac{(N_j+g_j-1)!}{N_j!\left(g_j-1\right)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j-\mu)/kT}-1}$
Microcanonical	$f_{mc}(q,p) = \frac{\delta\left(H(q,p)-E\right)}{\int dqdp\,\delta\left(H(q,p)-E\right)}$		
Canonical	$f_C(q,p) = \frac{\exp\left(-\beta H(q,p)\right)}{Z(N,V,T)}$	$Z(N,V,T) = \int dqdp\exp\left(-\beta H(q,p)\right)$	
Grand-canonical	$f_G(q,p) = \frac{\exp\left(\beta\left(\mu N - H\right)\right)}{\Xi\left(\mu,V,T\right)}$	$\Xi\left(\mu,V,T\right) = \sum_{N=0}^{\infty} z^N Z\left(N,V,T\right)$	

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2\left(\frac{\partial\ln Z}{\partial T}\right)_V$ Pressure $P = kT\left(\frac{\partial\ln Z}{\partial V}\right)_T$

Mathematical identities

$$\ln N! \approx N \ln N - N \qquad \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \qquad \frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$$

$$1 + y + y^2 + \ldots = \frac{1}{1-y} \qquad \int_{-\infty}^\infty dx e^{-x^2/\alpha} = \sqrt{\pi\alpha} \qquad \int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh(x) = \cosh(x) \qquad \mathrm{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh(x) = \sinh(x) \qquad \mathrm{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \qquad \frac{d}{dx}\coth(x) = -\mathrm{csch}^2(x)$$

QUESTION 1 (20 marks)

(a) A system of four distinguishable particles has allowed nondegenerate energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots$, and has a total energy $U = 6\varepsilon$. Tabulate all possible distributions of the particles among the allowed energy levels. Calculate the thermodynamic weight of each macrostate and the average occupation number of each of the energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, \ldots$.

(b) Tabulate the possible distributions if the particles are indistinguishable bosons and calculate the average occupation number of each level.

(c) Tabulate the possible distributions if the particles are fermions and the energy levels are nondegenerate and calculate the average occupation number of each level.

(d) In part (a), if energy level 3ε is missing, what is the average occupation of energy level 6ε?

The partition function is given by

$$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j / kT}$$

(e) If the density of states $g(\varepsilon)d\varepsilon = \left(4\sqrt{2}\pi V / h^3\right) m^{3/2} \varepsilon^{1/2} d\varepsilon$, approximate the sum by an integral and derive the result

$$Z = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V$$

(f) Calculate the internal energy and pressure for this system.

(g) Calculate the heat capacity at constant volume.

QUESTION 2 (20 marks)

(a) For a classical system of N particles with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \Phi(\mathbf{q}_1, \ldots, \mathbf{q}_N)$$

show that the canonical partition function is given by

$$Z = \int d\mathbf{p}_1 \ldots d\mathbf{p}_N \int d\mathbf{q}_1 \ldots d\mathbf{q}_N \exp(-\beta H) = (2\pi mkT)^{3N/2} \int d\mathbf{q}_1 \ldots d\mathbf{q}_N \exp(-\beta\Phi)$$

(b) For an ideal gas the potential is equal to zero everywhere (except at collisions). Write down the ideal gas canonical partition function.

(c) Use the canonical partition function to calculate the average internal energy and pressure of an ideal gas.

(d) The classical entropy is given by

$$S = \frac{\partial}{\partial T}(kT \ln Z).$$

Write the entropy as a function of T and V.

(e) Calculate the internal energy for the non-ideal gas partition function given in part (a). What does the result suggest about kinetic and potential contributions to the internal energy?

(f) Why is it difficult to calculate the pressure for the non-ideal gas in the same way we calculated it for the ideal gas?

QUESTION 3 (20 marks)

A dipole with magnetic moment μ in an external magnetic field $\mathbf{B}$ will experience a torque $\mathbf{N}$ given by

$$\mathbf{N} = \mu \times \mathbf{B}.$$

The magnetic potential energy ε is the work done to rotate the dipole from its zero energy position $\theta = \frac{\pi}{2}$

$$\varepsilon = \int_{\pi/2}^{\theta} N d\theta'$$

(a) Show that $\varepsilon = -\mu \cdot \mathbf{B} = -\mu_z B$.

The magnetic energy of an atom in quantum state m is $\varepsilon_m = -g\mu_B Bm$ where m is within the range $-J \le m \le J$. Here g is the Landé g-factor and $\mu_B = e\hbar/2m_e$ is the Bohr magneton. Each atom is in a localized position within a crystal and thus the atoms are distinguishable. Using Boltzmann statistics the probability of state m is

$$P_m = \frac{N_m}{N} = \frac{\exp(-\varepsilon_m/kT)}{Z}.$$

(b) Write down the partition function for this system.

The mean z-component of the magnetic moment is

$$\overline{\mu}_z = \sum_{m=-J}^{J} \mu_z P_m = \frac{1}{Z} \sum_{m=-J}^{J} g\mu_B m \exp(g\mu_B Bm/kT)$$

(c) Show that $\overline{\mu}_z$ can be written as the derivative of the logarithm of the partition function.

(d) If we let $\eta = g\mu_B B/kT$ and write $x = \exp(\eta) = \exp(g\mu_B B/kT)$, then show that the partition function can be written as

$$Z = \frac{\sinh\left(J + \frac{1}{2}\right)\eta}{\sinh(\eta/2)}$$

(e) Hence show that $\overline{\mu}_z = g\mu_B J B_J(\eta)$ and thus determine the Brillouin function $B_J(\eta)$ in its simplest form.

QUESTION 4 (20 marks)

For a system of fermions where the density of states is given by

$$g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\varepsilon^{1/2}d\varepsilon.$$

(a) Show that the Fermi energy at $T = 0$ is given by

$$\mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

The internal energy of a fermion gas is

$$U = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

(b) Explain the interplay between the numerator and denominator of the integrand in determining the value of the internal energy.

The electronic contribution to the internal energy is

$$U \approx \frac{3}{5}N\varepsilon_F\left(1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 - \ldots\right)$$

(c) Find and expression for the electronic heat capacity.

(d) Why is the electronic heat capacity much smaller than the heat capacity associated with other degrees of freedom?

QUESTION 5 (20 marks)

(a) A Bose gas at low temperature ($T < T_B$, where T_B is the Bose temperature) has an internal energy of

$$U = 0.77NkT\left(\frac{T}{T_B}\right)^{3/2}$$

determine the heat capacity at constant volume.

(b) As the result for the heat capacity is correct at zero temperature, we can calculate the entropy by integrating the heat capacity

$$S = \int_0^T \frac{C_V}{T'}\,dT'.$$

(c) Thus show that the Helmholtz function is given by

$$F = -0.51NkT\left(\frac{T}{T_B}\right)^{3/2}$$

(d) In the Bose temperature is given by

$$T_B = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612V}\right)^{2/3}$$

use the Helmholtz function to find the pressure.

(e) Hence show that $P = \dfrac{2U}{3V}$.

(f) Discuss possible connections between the theoretical Bose-Einstein condensation and the experimentally observed lambda transition between Helium I and Helium II.