THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE/JULY 2006

PHYS3020

Statistical Physics

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

All questions ARE of equal value

Candidates may not bring their own calculators.

The following materials will be provided by the Enrolment and Assesment Section: Calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

FORMULA SHEET

Boltzmann Entropy $S = k \ln W$

Statistics and Distributions

Boltzmann	$W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Maxwell-Boltzmann	$W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Fermi-Dirac	$W_{FD} = \prod_{j=1}^{n} \frac{g_j!}{N_j!(g_j - N_j)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$
Bose-Einstein	$W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} - 1}$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = NkT^2 \left(\frac{\partial \ln Z}{\partial T} \right)_V$ Pressure $P = NkT \left(\frac{\partial \ln Z}{\partial V} \right)_T$

Mathematical identities

$\ln N! \approx N \ln N - N$ $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$ $\frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$

$1 + y + y^2 + \ldots = \frac{1}{1-y}$ $\int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha}$ $\int_0^\infty d\varepsilon \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$

QUESTION 1 (20 marks)

(a) A system of three distinguishable particles has allowed nondegenerate energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, ...$, and has a total energy $U = 3\varepsilon$. Tabulate all possible distributions of the particles among the allowed energy levels. Calculate the thermodynamic weight of each macrostate and the average occupation number of each of the energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, ...$

(b) Tabulate the possible distributions if the particles are indistinguishable bosons and calculate the average occupation number of each level.

(c) Tabulate the possible distributions if the particles are fermions and the energy levels are nondegenerate and calculate the average occupation number of each level.

(d) Consider the dilute gas limit ($N_j << g_j$) for both Bose-Einstein and Fermi-Dirac statistics. Hence derive Maxwell-Boltzmann statistics

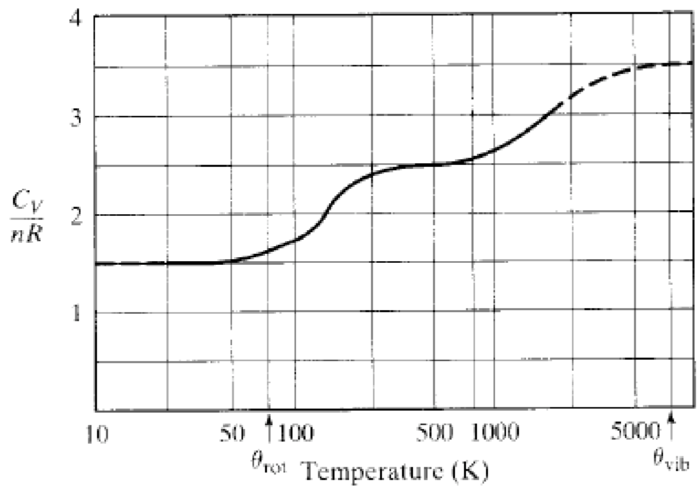
$$W_{MB} = \prod_j \frac{g_j^{N_j}}{N_j!}$$

How does this differ from Boltzmann statistics?

(e) If $x = (\varepsilon_j - \mu)/kT$, draw a graph of N_j/g_j versus x for Bose-Einstein, Fermi-Dirac and Maxwell-Boltzmann statistics.

QUESTION 2 (20 marks)

The figure below shows the experimental values of the heat capacity C_V/nR for hydrogen.



(a) Explain the qualitative behaviour of the heat capacity from both the classical and quantum approach. What are the deficiencies of the classical approach? How does the quantum mechanical approach remedy these deficiencies? Note that $\theta_{rot} = \hbar^2/2Ik$ and $\theta_{vib} = hv/k$.

(b) If the diatomic molecules are considered to be oscillators with energy levels $\varepsilon_j = \left(j + \frac{1}{2}\right)h\nu$, show that the partition function is given by

$$Z = \frac{e^{-\theta/2T}}{1 - e^{-\theta/T}}$$

(c) Derive the energy and the heat capacity for this system of oscillators.

(d) Discuss the behaviour of the heat capacity as $T \to 0$ and as $T \to \infty$.

(e) In reference to the heat capacity for hydrogen given in the graph above, how would this change if deuterium is used in place of normal hydrogen.

QUESTION 3 (20 marks)

(a) The classical grand canonical partition function is given by

$$\Xi(\mu,V,T)=\sum_{N=0}^{\infty}e^{\beta\mu N}Z(N,V,T)=\sum_{N=0}^{\infty}z^{N}Z(N,V,T).$$

where $z=\exp(\beta\mu)$ is the fugacity and $Z(N,V,T)$ is the canonical partition function. Show that the average number of particles $\langle N\rangle$

$$\langle N\rangle=\frac{1}{\Xi}\sum_{N=0}^{\infty}Nz^{N}Z(N,V,T)=z\frac{\partial}{\partial z}\ln\Xi.$$

That is, if the first equality defines the average $\langle N\rangle$, derive the second equality.

(b) If $\Delta N=N-\langle N\rangle$, show that $\langle\Delta N^{2}\rangle=\langle N^{2}\rangle-\langle N\rangle^{2}$.

(c) Derive the following expression for the fluctuations in the average number of particles,

$$\langle\Delta N\rangle^{2}=z^{2}\frac{\partial^{2}}{\partial z^{2}}\ln\Xi+z\frac{\partial}{\partial z}\ln\Xi.$$

(d) The thermodynamic potential for the grand partition function is

$$\Omega(T,V,\mu)=-kT\ln\Xi=U-TS-\mu N$$

What can be said about the behaviour of the relative fluctuations in the number of particles $\sqrt{\langle\Delta N^{2}\rangle}\big/\langle N\rangle$.

QUESTION 4 (20 marks)

(a) Photons in a cavity obey Bose-Einstein statistics. If the number of quantum states with frequencies in the range v to $v + dv$ is

$$g(v)dv = \frac{8\pi V}{c^3} v^2 dv$$

show that the energy density is

$$u(v)dv = \frac{8\pi h V}{c^3}\left(\frac{v^3 dv}{e^{hv/kT} - 1}\right)$$

(b) Find the total energy density (energy per unit volume) by integrating over wavelength ($\lambda = c/v$). If the total energy density can be written as

$$\frac{U}{V} = aT^4$$

find the explicit expression for the constant a.

(c) Find the wavelength for which the $u(\lambda)$ is a maximum.

QUESTION 5 (20 marks)

(a) For a system of fermions where the density of states is

$$g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\varepsilon^{1/2}d\varepsilon,$$

show that the chemical potential at $T = 0$ (that is the Fermi energy $\varepsilon_F = \mu(0)$) is given by

$$\mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}.$$

(b) For $T \neq 0$ the chemical potential can be approximated by

$$\mu(T) = \mu(0) - \alpha T^2$$

Find the change in the Fermi-Dirac distribution as a function of T.

(c) Assuming this new distribution can be approximated by

$$f_T(\varepsilon) \approx \exp\left(-\frac{\alpha T}{k}\right) f_{T=0}(\varepsilon)$$

Derive an expression for the proportion of fermions below the Fermi energy $\mu(0)$.