THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2011

PHYS3020 /9583

Statistical Physics

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

All questions ARE of equal value

Students are required to supply their own university approved calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

FORMULA SHEET

Boltzmann Entropy $S = k \ln W$

Statistics and Distributions

Boltzmann	$W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Maxwell-Boltzmann	$W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Fermi-Dirac	$W_{FD} = \prod_{j=1}^{n} \frac{g_j!}{N_j!(g_j - N_j)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$
Bose-Einstein	$W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} - 1}$

Microcanonical $$f_{mc}(q,p) = \frac{\delta(H(q,p) - E)}{\int dq dp\, \delta(H(q,p) - E)}$$

Canonical $$f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)} \qquad Z(N,V,T) = \int dq dp \exp(-\beta H(q,p))$$

Grand-canonical $$f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)} \qquad \Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T)$$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2 \left(\frac{\partial \ln Z}{\partial T} \right)_V$ Pressure $P = kT \left(\frac{\partial \ln Z}{\partial V} \right)_T$

Mathematical identities

$$\ln N! \approx N \ln N - N \qquad \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \qquad \frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$$

$$1 + y + y^2 + \ldots = \frac{1}{1-y} \qquad \int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha} \qquad \int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh(x) = \cosh(x) \qquad \operatorname{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh(x) = \sinh(x) \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \qquad \frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$$

$$\int_{-\infty}^{\infty} \frac{e^y dy}{\left(e^y + 1\right)^2} = 1 \qquad \int_{-\infty}^{\infty} \frac{y^2 e^y dy}{\left(e^y + 1\right)^2} = \frac{\pi^2}{3}$$

QUESTION 1 (20 marks)

(a) Consider a system in which the allowed energy levels $0, \varepsilon, 2\varepsilon, 3\varepsilon, 4\varepsilon, ...$ are nondegenerate ($g_j = 1$). If the system has 4 distinguishable particles and a total energy of $U = 8\varepsilon$, tabulate the 15 possible distributions of the particles among the energy levels and calculate the thermodynamic weight of each macrostate.

(b) Explain how this tabulation changes if the particles are indistinguishable bosons.

(c) Construct the same table for indistinguishable fermions.

(d) Calculate the average occupancy of each energy level for the fermion system.

(e) How many ways can the energy of the fermion system be increased to a total energy of 9ε? How does this compare with the number of ways of increasing the energy of the boson system?

(f) How many macrostates are possible for 4 fermions with energy 9ε? Calculate the average occupancy of each energy level in the system with energy 9ε.

(g) Calculate the change in entropy for the fermion system when the energy is increased from 8ε to 9ε.

(h) If this energy change takes place isothermally, calculate the temperature.

QUESTION 2 (20 marks)

(a) Show that the thermodynamic probability

$$W = \prod_{j=1}^{n} \frac{g_j(g_j - a)(g_j - 2a)\ldots(g_j - (N_j - 1)a)}{N_j!}$$

Reduces to Maxwell-Boltzmann statistics when $a = 0$, to Fermi-Dirac statistics when $a = 1$, and to Bose-Einstein statistics when $a = -1$.

(b) If the density of states $g(\varepsilon)d\varepsilon = \left(4\sqrt{2}\pi V/h^3\right)m^{3/2}\varepsilon^{1/2}d\varepsilon$, the sum can be approximated by an integral to give the partition function

$$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} V$$

Calculate the internal energy and pressure for this system using

$$U = NkT^2\left(\frac{\partial \ln Z}{\partial T}\right)_V \quad \text{and} \quad P = NkT\left(\frac{\partial \ln Z}{\partial V}\right)_T .$$

(c) If the Helmholtz function $F = -NkT(\ln(Z/N) + 1)$, show that the entropy of the system is

$$S = Nk\left(\frac{3}{2}\ln T - \ln\left(\frac{N}{V}\right)\right) + S_0$$

and find the value of the constant S_0.

d) If the partition function of a system that obeys Maxwell-Boltzmann statistics is given by $Z = aV^{\alpha}T^{\beta}$, where a is a constant, find the relationship between the pressure P and internal energy per unit volume U/V.

QUESTION 3 (20 marks)

(a) The Gibbs ensemble is a collection of identical systems with different initial conditions all in contact with the same reservoir. If the probability of a system being in state j is p_j then in the ensemble $n_j = p_j n$ systems are in state j. If the thermodynamic weight for the ensemble is

$$w_n = \frac{n!}{n_1! n_2! \ldots n_j! \ldots}$$

derive the Gibbs entropy.

(b) For a classical system of N particles with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \Phi(\mathbf{q}_1, \ldots, \mathbf{q}_N)$$

show that the canonical partition function is given by

$$Z = \int d\mathbf{p}_1 .. d\mathbf{p}_N \int d\mathbf{q}_1 .. d\mathbf{q}_N \exp(-\beta H) = (2\pi mkT)^{3N/2} \int d\mathbf{q}_1 .. d\mathbf{q}_N \exp(-\beta\Phi) .$$

(c) For an ideal gas the potential is equal to zero everywhere (except at collisions). Ignoring collisions, find the classical canonical partition function for the ideal gas.

(d) Show that the average internal energy is obtained from the following derivative of the partition function

$$\langle U \rangle = -\left(\frac{\partial \ln Z(N,V,T)}{\partial \beta} \right)_V$$

(e) Show that a measure of the average fluctuation in the internal energy $\Delta U = U - \langle U \rangle$ can be estimated by calculating the mean square fluctuation $\langle \Delta U^2 \rangle = \langle U^2 \rangle - \langle U \rangle^2$. Why not average $U - \langle U \rangle$?

(f) How can the classical partition function be corrected so that it agrees with the quantum result?

QUESTION 4 (20 marks)

(a) The partition function z for a *single oscillator* or photon (ignoring the zero-point energy) is given by

$$\ln z = -\ln\left(1 - e^{-h\nu/kT}\right),$$

The number of single particle (photon) states in a volume V in the frequency range ν to $\nu + d\nu$ is

$$g(\nu)d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu$$

The logarithm of the partition function Z for the photon gas is the sum over single oscillator states

$$\ln Z = \int_0^\infty g(\nu)d\nu \ln z$$

Show that integration by parts (with $x \equiv h\nu/kT$) leads to the equation

$$\ln Z = \frac{8\pi V}{3}\left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

(b) Hence show that

$$\ln Z = \frac{8\pi^5}{45}\left(\frac{kT}{hc}\right)^3 V$$

(c) Find the total internal energy U and the chemical potential for this system. Note that

$$\mu = -kT\left(\frac{\partial \ln Z}{\partial N}\right)_{V,T}.$$

(d) Use $-kT\ln Z = U - TS$ to find the entropy for the system.

(e) Find the pressure for the system.

(f) Show that for photons $P = \frac{1}{3}(U/V)$ which is different to an ideal gas of molecules.

QUESTION 5 (20 marks)

(a) For a system of fermions where the density of states is given by

$$g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\varepsilon^{1/2}d\varepsilon.$$

Show that the Fermi energy at $T = 0$ is given by

$$\varepsilon_F = \mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

(b) The internal energy of a fermion gas is

$$U = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

Explain the method used to calculate the internal energy from this integral.

(c) If the electronic contribution to the internal energy is

$$U = \tfrac{3}{5}N\varepsilon_F\left(1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 - \ldots\right),$$

where ε_F is the Fermi energy and $T_F(V)$ is the Fermi temperature, find an expression for the electronic heat capacity. Note that the Fermi temperature is given by

$$T_F = \frac{h^2}{2mk}\left(\frac{3N}{8\pi V}\right)^{2/3}.$$

(d) If the internal energy can be written as an infinite sum

$$U = \tfrac{3}{5}NkT_F\sum_{i=0}^{\infty}c_i\left(\frac{T}{T_F}\right)^{2i},$$

show that the entropy is given by

$$S = \int_0^T \frac{C_V}{T'} dT' = \tfrac{3}{5} Nk \sum_{i=1}^{\infty} \frac{2i}{2i-1} c_i \left(\frac{T}{T_F}\right)^{2i-1} .$$

Hence show that the Helmholtz function is

$$F = U - TS = \tfrac{3}{5} NkT_F \left\{ 1 - \sum_{i=1}^{\infty} \frac{c_i}{2i-1} \left(\frac{T}{T_F}\right)^{2i} \right\}$$

and that the relation $P = \tfrac{2}{3}(U/V)$ is exact for the fermion gas.