THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2009

PHYS3020

Statistical Physics

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

All questions ARE of equal value

Students are required to supply their own university approved calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

FORMULA SHEET

Boltzmann Entropy $S = k \ln W$

Statistics and Distributions

Boltzmann	$W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Maxwell-Boltzmann	$W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$	$\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$	$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$
Fermi-Dirac	$W_{FD} = \prod_{j=1}^{n} \frac{g_j!}{N_j!(g_j - N_j)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$
Bose-Einstein	$W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$		$\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} - 1}$

Microcanonical $f_{mc}(q,p) = \frac{\delta(H(q,p) - E)}{\int dq dp \delta(H(q,p) - E)}$

Canonical $f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)}$ $Z(N,V,T) = \int dq dp \exp(-\beta H(q,p))$

Grand-canonical $f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)}$ $\Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T)$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2 \left(\frac{\partial \ln Z}{\partial T} \right)_V$ Pressure $P = kT \left(\frac{\partial \ln Z}{\partial V} \right)_T$

Mathematical identities

$$\ln N! \approx N \ln N - N \qquad \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \qquad \frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$$

$$1 + y + y^2 + \ldots = \frac{1}{1-y} \qquad \int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha} \qquad \int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \frac{d}{dx}\sinh(x) = \cosh(x) \qquad \mathrm{csch}(x) = \frac{1}{\sinh(x)}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \frac{d}{dx}\cosh(x) = \sinh(x) \qquad \mathrm{sech}(x) = \frac{1}{\cosh(x)}$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)} \qquad \frac{d}{dx}\tanh(x) = \mathrm{sech}^2(x)$$

$$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} \qquad \frac{d}{dx}\coth(x) = -\mathrm{csch}^2(x)$$

QUESTION 1 (20 marks)

(a) Show that the thermodynamic probability

$$W = \prod_{j=1}^{n} \frac{g_j(g_j - a)(g_j - 2a)...(g_j - (N_j - 1)a)}{N_j!}$$

reduces to Maxwell-Boltzmann statistics when $a = 0$, to Fermi-Dirac statistics when $a = 1$, and to Bose-Einstein statistics when $a = -1$.

(b) Find the thermodynamic probability in the dilute gas limit $g_j >> N_j$.

(c) If the distribution is Maxwell-Boltzmann, calculate the entropy and the Helmholtz function.

(d) If the system has only two energy levels 0 and ε_1, with degeneracies g_0 and g_1, write down the entropy and Helmholtz function for the system. Note that

$$Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$$

(e) Find the Helmholtz function in the limit as $T \to 0$.

(f) Find an equation for the pressure of this system and explain the physical nature of the derivative that is needed.

QUESTION 2 (20 marks)

An assembly of one-dimensional harmonic oscillators are loosely coupled so that they each oscillate independently in one of the energy levels $\varepsilon_j = (j + \frac{1}{2})h\nu$, where $j = 0,1,2,3,....$

(a) In the dilute gas approximation $g_j >> N_j$, derive the partition function Z for this system using the characteristic temperature $\theta = h\nu/k$.

(b) Calculate the occupation number of the j^{th} energy level, that is find N_j/N.

(c) Why can Boltzmann statistics for distinguishable particles be used in this case?

(d) Use the partition function show that the total energy is

$$U = Nk\theta\left(\frac{1}{2} + \frac{1}{e^{\theta/T} - 1}\right)$$

(e) What is the form for the energy as $T \to 0$, and as $T \to \infty$?

(f) Outline the model in Einstein's theory of the heat capacity of a solid and write down an equation for the energy as a function of temperature.

(g) Outline the physical picture or beginning assumptions in Debye's model of heat capacity. In particular, how does it differ from the Einstein model?

(h) Show that the distribution of frequencies in the Debye model of $3N$ oscillators is given by

$$g(\nu)d\nu = \frac{9N}{\nu_m^3}\nu^2 d\nu$$

QUESTION 3 (20 marks)

(a) For a classical system of N particles with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \Phi(\mathbf{q}_1,...,\mathbf{q}_N)$$

show that the canonical partition function is given by

$$Z = \int d\mathbf{p}_1..d\mathbf{p}_N \int d\mathbf{q}_1..d\mathbf{q}_N \exp(-\beta H) = (2\pi mkT)^{3N/2} \int d\mathbf{q}_1..d\mathbf{q}_N \exp(-\beta\Phi).$$

(b) For an ideal gas the potential is equal to zero everywhere (except at collisions). Write down the ideal gas canonical partition function.

(c) Use the canonical partition function for the ideal gas to calculate the average internal energy and pressure.

(d) Show that fluctuations in the internal energy in the canonical ensemble are given by

$$\frac{\partial^2}{\partial\beta^2} \ln Z = \langle U^2 \rangle - \langle U \rangle^2.$$

(e) Show that the mean square fluctuation in the internal energy $\langle \Delta U^2 \rangle = \langle U^2 \rangle - \langle U \rangle^2$ in the canonical ensemble is determined by the heat capacity at constant volume.

(f) The classical grand canonical partition function is given by

$$\Xi(z,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T).$$

Where the fugacity z is related to the chemical potential by $z = \exp(\beta\mu)$ and $Z(N,V,T)$ is the canonical partition function. Show that the average number of particles $\langle N \rangle$ given by

$$\langle N \rangle \equiv \frac{1}{\Xi} \sum_{N=0}^{\infty} N z^N Z(N,V,T)$$

can be written as a derivative of the grand canonical partition function.

(g) Write the average fluctuation in N, that is, the average of $\Delta N^2 = \left(N - \langle N \rangle\right)^2$, as a derivative of the grand canonical partition function $\Xi(z,V,T)$.

QUESTION 4 (20 marks)

(a) A Bose gas at low temperature ($T < T_B$, where T_B is the Bose temperature) has an internal energy of

$$U = 0.77NkT\left(\frac{T}{T_B}\right)^{3/2}$$

determine the heat capacity at constant volume.

(b) As the result for the heat capacity is correct at zero temperature, calculate the entropy using

$$S = \int_0^T \frac{C_V}{T'}\,dT'.$$

(c) Thus show that the Helmholtz function is given by

$$F = -0.51NkT\left(\frac{T}{T_B}\right)^{3/2}.$$

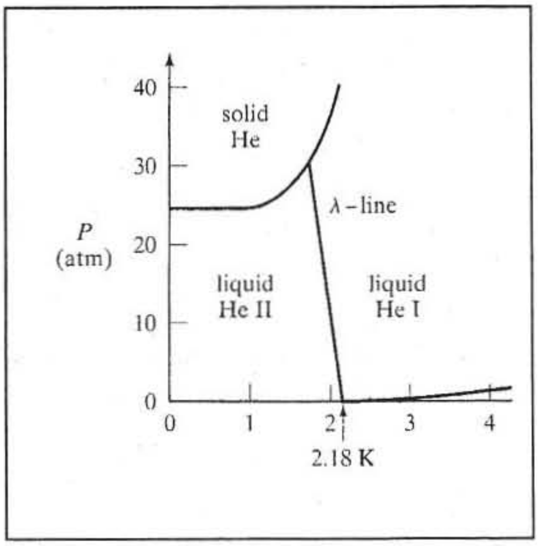
(d) If the Bose temperature is given by

$$T_B = \frac{h^2}{2\pi mk}\left(\frac{N}{2.612V}\right)^{2/3}$$

use the Helmholtz function to find the pressure.

(e) Hence show that $P = \dfrac{2U}{3V}$.

(f) Discuss possible connections between the theoretical Bose-Einstein condensation and the experimentally observed lambda transition between Helium I and Helium II.



QUESTION 5 (20 marks)

(a) For a system of fermions where the density of states is given by

$$g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon .$$

Show that the Fermi energy at $T = 0$ is given by

$$\varepsilon_F = \mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

(b) The internal energy of a fermion gas is

$$U = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2} d\varepsilon}{e^{(\varepsilon-\mu)/kT} + 1}$$

Explain the interplay between the numerator and denominator of the integrand in determining the value of the internal energy.

(c) The fermionic contribution to the internal energy is

$$U \approx \frac{3}{5} N\varepsilon_F \left(1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 - \ldots\right)$$

Find an expression for the heat capacity.

(d) Integrate the entropy from $TdS = C_V dT$ to obtain the entropy, and hence, the Helmholtz function.

$$F = NkT_F\left[\frac{3}{5} - \frac{\pi^2}{4}\left(\frac{T}{T_F}\right)^2 + \ldots\right],$$

where the Fermi temperature is $T_F = \dfrac{h^2}{2mk}\left(\dfrac{3N}{8\pi V}\right)^{2/3}$.

(e) Calculate the fermionic contribution to the pressure for a gas of electrons.

(f) If a white dwarf star consists of alpha particles and a degenerate electron gas find the electronic contribution to the internal energy. If the gravitational potential energy $U_{grav} = -\frac{b}{R}$ where R is the radius of the star, explain how a minimum energy results in a stable radius for the star.