THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2008

PHYS3020

Statistical Physics

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

All questions ARE of equal value

Candidates may not bring their own calculators.

The following materials will be provided by the Enrolment and Assesment Section: Calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

PPJ

FORMULA SHEET

Boltzmann Entropy $S = k \ln W$

Statistics and Distributions

Boltzmann $W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Maxwell-Boltzmann $W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!}$ $\frac{N_j}{g_j} = \frac{N}{Z} e^{-\varepsilon_j/kT}$ $Z = \sum_{j=1}^{n} g_j e^{-\varepsilon_j/kT}$

Fermi-Dirac $W_{FD} = \prod_{j=1}^{n} \frac{g_j!}{N_j!(g_j - N_j)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} + 1}$

Bose-Einstein $W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!}$ $\frac{N_j}{g_j} = \frac{1}{e^{(\varepsilon_j - \mu)/kT} - 1}$

Microcanonical $f_{mc}(q,p) = \frac{\delta(H(q,p) - E)}{\int dq dp\, \delta(H(q,p) - E)}$

Canonical $f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)}$ $Z(N,V,T) = \int dq dp \exp(-\beta H(q,p))$

Grand-canonical $f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)}$ $\Xi(\mu,V,T) = \sum_{N=0}^{\infty} z^N Z(N,V,T)$

Thermodynamic Potentials

Internal energy	U	$dU = TdS - PdV$
Enthalpy	$H = U + PV$	$dH = TdS + VdP$
Helmholtz function	$F = U - TS$	$dF = -SdT - PdV$
Gibbs function	$G = U - TS + PV$	$dG = -SdT + VdP$

Statistical Mechanics Canonical Ensemble

Internal energy $U = kT^2 \left(\frac{\partial \ln Z}{\partial T} \right)_V$ Pressure $P = kT \left(\frac{\partial \ln Z}{\partial V} \right)_T$

Mathematical identities

$\ln N! \approx N \ln N - N$ $\qquad$ $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$ $\qquad$ $\frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96$

$1 + y + y^2 + \ldots = \frac{1}{1-y}$ $\qquad$ $\int_{-\infty}^{\infty} dx e^{-x^2/\alpha} = \sqrt{\pi\alpha}$ $\qquad$ $\int_0^\infty d\varepsilon\, \varepsilon^{1/2} e^{-\varepsilon/\alpha} = \frac{\alpha}{2}\sqrt{\pi\alpha}$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$ $\qquad$ $\frac{d}{dx}\sinh(x) = \cosh(x)$ $\qquad$ $\operatorname{csch}(x) = \frac{1}{\sinh(x)}$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\qquad$ $\frac{d}{dx}\cosh(x) = \sinh(x)$ $\qquad$ $\operatorname{sech}(x) = \frac{1}{\cosh(x)}$

$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ $\qquad$ $\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$ $\qquad$ $\frac{d}{dx}\tanh(x) = \operatorname{sech}^2(x)$

$\coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ $\qquad$ $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$ $\qquad$ $\frac{d}{dx}\coth(x) = -\operatorname{csch}^2(x)$

QUESTION 3 (20 marks)

(a) For a classical system of N particles with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} + \Phi(\mathbf{q}_1,...,\mathbf{q}_N)$$

show that the canonical partition function is given by

$$Z = \int d\mathbf{p}_1..d\mathbf{p}_N \int d\mathbf{q}_1..d\mathbf{q}_N \exp(-\beta H) = (2\pi mkT)^{3N/2} \int d\mathbf{q}_1..d\mathbf{q}_N \exp(-\beta\Phi)$$

(b) For an ideal gas the potential is equal to zero everywhere (except at collisions). Write down the ideal gas canonical partition function.

(c) Use the canonical partition function to calculate the average internal energy and pressure of an ideal gas.

(d) If the classical average of an arbitrary phase variable X is given by

$$\langle X \rangle = \frac{\int dqdpXe^{-\beta H}}{\int dqdpe^{-\beta H}},$$

show that

$$\frac{\partial}{\partial \beta} \ln Z = -U$$

and

$$\frac{\partial^2}{\partial \beta^2} \ln Z = \langle U^2 \rangle - \langle U \rangle^2.$$

(e) Show that the mean square fluctuation in the internal energy $\langle \Delta U^2 \rangle = \langle U^2 \rangle - \langle U \rangle^2$ in the canonical ensemble is determined by the heat capacity at constant volume.

QUESTION 4 (20 marks)

(a) Photons in a cavity obey Bose-Einstein statistics. If the number of quantum states with frequencies in the range ν to $\nu + d\nu$ is

$$g(\nu)d\nu = \frac{8\pi V}{c^3}\nu^2 d\nu$$

show that the energy density is

$$u(\nu)d\nu = \frac{8\pi h V}{c^3}\left(\frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}\right)$$

(b) Find the total energy density (energy per unit volume) by integrating over wavelength ($\lambda = c/\nu$). If the total energy density can be written as

$$\frac{U}{V} = aT^4$$

find the explicit expression for the constant a.

(c) Explain how the energy density as a function of wavelength given above (Planck's law) is related to the Rayleigh-Jeans law $u(\lambda)d\lambda \approx 8\pi kTVd\lambda/\lambda^4$, and to Wien's law

$$u(\lambda)d\lambda \approx V\left(\frac{8\pi hc}{\lambda^5}\right)e^{-hc/\lambda kT}d\lambda.$$

QUESTION 5 (20 marks)

(a) For a system of fermions where the density of states is given by

$$g(\varepsilon)d\varepsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\varepsilon^{1/2}d\varepsilon.$$

Show that the Fermi energy at $T = 0$ is given by

$$\mu(0) = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{2/3}$$

(b) The internal energy of a fermion gas is

$$U = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\int_0^\infty \frac{\varepsilon^{3/2}d\varepsilon}{e^{(\varepsilon-\mu)/kT}+1}$$

Explain the interplay between the numerator and denominator of the integrand in determining the value of the internal energy.

(c) The electronic contribution to the internal energy is

$$U \approx \frac{3}{5}N\varepsilon_F\left(1+\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2 - \ldots\right)$$

Find an expression for the electronic heat capacity.

(d) The internal energy can be written as an infinite sum with a set of undetermined coefficients

$$U = \frac{3}{5}NkT_F\sum_{i=0}^{\infty}a_i\left(\frac{T}{T_F}\right)^{2i},$$

where

$$T_F = \frac{h^2}{2mk}\left(\frac{3N}{8\pi V}\right)^{2/3}.$$

The dependence on T is explicit and T_F is a function of V. Show that the entropy is given by

$$S = \int_0^T \frac{1}{T'} \frac{\partial U}{\partial T'} dT' = \tfrac{3}{5} NkT_F \sum_{i=1}^{\infty} \frac{2i}{2i-1} a_i \left(\frac{T^{2i-1}}{T_F^{2i}} \right).$$

Show that the Helmholtz function is

$$F = U - TS = \tfrac{3}{5} NkT_F \left\{ 1 - \sum_{i=1}^{\infty} \frac{a_i}{2i-1} \left(\frac{T}{T_F} \right)^{2i} \right\}.$$

Hence, or otherwise, show that the relation $P = \frac{2}{3}(U/V)$ is exact for the fermion gas.