THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION: JUNE 2013

PHYS3011 & PHYS3230
Electrodynamics

Time allowed – 2 hours
Total number of questions – 4
Answer ALL 4 questions
All questions are of equal value

Candidates must supply their own,
university-approved calculator.

All answers must be written in ink.
Except where they are expressly required,
pencils may only be used for drawing,
sketching, or graphical work.

Candidates may keep this question paper.

Question 1

A parallel-plate capacitor is made with the sides sealed and the space between the plates half filled with a liquid of relative permittivity ϵ_r. The capacitor is charged and isolated.

(a) Find an expression for the ratio of the potential difference between the plates when they are horizontal, V_H, to that when they are vertical, V_V.

(b) Hence find an expression for the ratio of the stored energy in the two cases, if the capacitor is charged and then isolated while it is tilted from one orientation to the other.

Question 2

What is the maximum power that an air-filled wave guide can transmit in the TE_{01} mode if the transverse dimensions of the guide are $a = 1$ cm, $b = 2$ cm and the frequency is 10^{10} Hz ?
Take the breakdown field in air to be 30 $\mathrm{kV\,cm^{-1}}$.

Question 3

An air-spaced coaxial cable has an inner conductor of radius 0.25 cm and an outer conductor of radius 0.75 cm. The inner conductor is at a potential of +8 kV with respect to the grounded outer conductor.
(a) Find the charge per metre on the inner conductor

(b) Find the electric field strength at r=0.5 cm.

Question 4

The electric field $\mathbf{E}$ from an oscillating electric dipole, $p_0 \cos \omega t$, at the origin and aligned along the z-axis, is given by:

$$\mathbf{E}(r,\theta,t) = \frac{-\mu_0\, p_0\, \omega^2 \sin\theta}{4\pi r} \cos\{\omega(t - r/c)\}\, \hat{\boldsymbol{\theta}}$$

where r is large compared to the size of the dipole, and to the wavelength of the radiation, so that the plane-wave approximation can be used.

(a) Write down the corresponding expression for $\mathbf{B}$. (Note that the direction of propagation, $\hat{\mathbf{k}}$, is along $\hat{\mathbf{r}}$.)

(b) Show that the Poynting vector, $\mathbf{S}$, for this radiation is

$$\mathbf{S} = \left(\frac{\mu_0 p_0^2 \omega^4}{16\pi^2 c}\right) \frac{\sin^2\theta}{r^2} \cos^2\{\omega(t - r/c)\}\, \hat{\mathbf{r}}$$

(c) and find the average total power radiated by the dipole.

End of Exam

Useful Formulae: PHYS3011/PHYS3230

$\epsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1} $\quad \mu_0 = 4\pi \times 10^{-7}$ Hm^{-1} $\quad c = 3 \times 10^8$ ms^{-1}

Volume element = $dx\,dy\,dz \;=\; r^2 \sin\theta\, dr\, d\theta\, d\phi$

Surface area of sphere = $4\pi r^2$ Volume of sphere = $\frac{4}{3}\pi r^3$

Divergence Theorem: $\int_V \nabla\cdot\mathbf{A}\, dV = \int_S \mathbf{A}\cdot d\mathbf{S}$ (S is the surface enclosing V)

Stokes' Theorem: $\int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S} = \oint_L \mathbf{A}\cdot d\mathbf{l}$ (L is the curve bounding S)

Vector identity: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$

So: $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\,\mathbf{E}$

Also: $\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$

∇^2 in spherical polar coordinates:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Electrostatics:

Charge conservation: $I = -\frac{dq}{dt}$

$q = \int_V \rho\, dV$ (charge density) $\qquad I = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{J}\, dV$ (current density)

$\therefore\ \nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$ (equation of continuity)

E field defined by: $\mathbf{F}_E = q\mathbf{E}$

Coulomb's Law: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}$

Gauss's Law: $\Phi_E = \int_S \mathbf{E}\cdot d\mathbf{S} = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\int_V \rho\, dV$

$\therefore\ \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$ (Gauss's Law)

[Also, since **E** is conservative in electro*statics*: $\nabla\times\mathbf{E} = 0$]

Magnetism:

B field defined by: $\mathbf{F}_B = q\mathbf{v}\times\mathbf{B}$ ie $d\mathbf{F} = dq\,\mathbf{v}\times\mathbf{B} = I d\mathbf{l}\times\mathbf{B}$

No magnetic monopoles: $\nabla\cdot\mathbf{B} = 0$

Biot-Savart Law: $d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I' d\mathbf{l}' \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{dq'}{r^2}\mathbf{v}'\times\hat{\mathbf{r}}$

Ampère's Law: $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 (N) I = \mu_0 \int_S \mathbf{J}\cdot d\mathbf{S}$

(where I = current enclosed)

$\therefore\ \nabla\times\mathbf{B} = \mu_0\mathbf{J}$ (Ampère's Law)

Faraday's Law: $\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$

$\therefore\ \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ (Faraday's Law)

Dielectric materials:

$\mathbf{P} = \chi\epsilon_0\mathbf{E} \qquad \mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = (1+\chi)\epsilon_0\mathbf{E} = \epsilon_r\epsilon_0\mathbf{E}$

E field (and potential difference) is reduced by a factor ϵ_r in the bulk.

Energy density, $u = \frac{1}{2}\epsilon_r\epsilon_0 E^2$ per unit volume $= \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$

Gauss's Law for **D** : $\int \mathbf{D}\cdot d\mathbf{S} = q_{free} \qquad \nabla\cdot\mathbf{D} = \rho_{free}$

At a boundary, $E_{\parallel}$ and V are continuous ($D_{\perp}$ is continuous.)

Cavities in dielectrics: $\mathbf{E}_{\text{local}} = \mathbf{E}_{\text{bulk}}$ for a needle-shaped cavity;

$\mathbf{E}_{\text{local}} = \mathbf{E}_{\text{bulk}} + \mathbf{P}/\epsilon_0$ for a disc-shaped cavity;

$\mathbf{E}_{\text{local}} = \mathbf{E}_{\text{bulk}} + \mathbf{P}/3\epsilon_0$ for a spherical cavity.

Clausius-Mossotti equation: $\frac{n\alpha}{3\epsilon_0} = \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)$

Capacitance:

stored charge $Q = C\Delta V$ [C]
stored energy $U = \frac{1}{2}Q\Delta V = \frac{1}{2}C(\Delta V)^2$ or $\frac{1}{2}Q^2/C$ [J]

capacitance of parallel-plate capacitor is $C = \epsilon_r\epsilon_0 A/d$ [F]

capacitance of isolated sphere is $C = 4\pi\epsilon_r\epsilon_0 R$ [F]

DC Circuits:

Ohm's Law: $\Delta V = IR$ resistance, $R = \rho l/A$ [Ω]

Kirchhoff's Laws: (1) $\Sigma I = 0$ at a junction
(2) $\Sigma \mathcal{E} - \Sigma IR = 0$ around each loop

Joule heating: power dissipated, $P = I\Delta V = I^2 R = (\Delta V)^2/R$ [W]

Ohm's law: $\mathbf{J} = \sigma \mathbf{E}$ power dissipated/unit volume $= \mathbf{J}\cdot\mathbf{E} = \sigma E^2$

Magnetic media:

$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu_0(1+\chi_m)\mathbf{H} = \mu_r\mu_0\mathbf{H}$ *ie* $\mathbf{H} = \dfrac{\mathbf{B}}{\mu_0} - \mathbf{M}$

$\nabla\cdot\mathbf{B} = 0$, so $\nabla\cdot\mathbf{H} + \nabla\cdot\mathbf{M} = 0$

Ampère's law becomes: $\nabla\times\mathbf{H} = \mathbf{J}_{free}$

At a boundary, $B'_\perp = B_\perp$ and $H'_\parallel = H_\parallel$

Inductance:

Mutual inductance: $\Phi_1 = L_{12} I_2$, $\Phi_2 = L_{12} I_1$, Self inductance: $\Phi = LI$

Self Inductance of a solenoid: $L = \mu_r\mu_0 \dfrac{N^2}{\ell} A$ magnetic energy: $U = \frac{1}{2} L I^2$

Energy density in magnetic field: $u = \frac{1}{2}\dfrac{B^2}{\mu_r\mu_0} = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$

Maxwell's Equations

In a vacuum:

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Lorentz force law: $\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})$

Maxwell's equations in dielectric and magnetic media:

$$\nabla\cdot\mathbf{D} = \rho \qquad \nabla\times\mathbf{E} = -\dot{\mathbf{B}}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad \nabla\times\mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}$$

EM Waves:

Wave equation for $\mathbf{E}$ in free space: $\nabla^2\mathbf{E} = \mu_0\epsilon_0\dfrac{\partial^2\mathbf{E}}{\partial t^2}$ *ie* $c = 1/\sqrt{\mu_0\epsilon_0}$

(in a medium: $v = 1/\sqrt{\mu_r\mu_0\epsilon_r\epsilon_0} = c/n$, n = refractive index)

Solution: $E = E_0\sin(kx - \omega t)$ for monochromatic wave travelling in +ve x-direction.

$\mathbf{E}$, $\mathbf{B}$ and the direction of propagation $\hat{\mathbf{k}}$ are mutually perpendicular:

$\hat{\mathbf{k}}\cdot\mathbf{E} = 0$ $\hat{\mathbf{k}}\cdot\mathbf{B} = 0$ $c\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}$ $\hat{\mathbf{E}}\times\hat{\mathbf{B}} = \hat{\mathbf{k}}$

The direction of $\mathbf{E}$ is the direction of polarization of the E-M wave.

Impedance of free space, $Z_0 = \dfrac{|E|}{|H|} = \sqrt{\dfrac{\mu_0}{\epsilon_0}} = 377\,\Omega$

Poynting vector: $\mathbf{N}$ (or $\mathbf{S}$) $= \mathbf{E}\times\mathbf{H} = \dfrac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = E^2/Z_0 = H^2 Z_0$

NB: Wave number, $k = \dfrac{2\pi}{\lambda}$ Angular frequency, $\omega = 2\pi f$

Phase velocity, $v_p = f\lambda = \dfrac{\omega}{k}$ Group velocity $v_g = \dfrac{d\omega}{dk}$

Malus' Law: $I(\theta) = I(0)\cos^2\theta$ for polarizers at relative angle θ.

Reflection and Refraction at interface between two dielectrics

Reflection: $\theta_r = \theta_i$

Refraction: $n_2\sin\theta_t = n_1\sin\theta_i$ (Snell's Law)

Critical angle: $\sin\theta_i = n_2/n_1$ if $n_1 > n_2$

Fresnel Equations ($\mu_r = 1$)

For $\mathbf{E}$ *parallel to the plane of incidence*

$$r_\parallel = \frac{E_r}{E_i} = \frac{(n_2/n_1)\cos\theta_i - \cos\theta_t}{(n_2/n_1)\cos\theta_i + \cos\theta_t} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$t_\parallel = \frac{E_t}{E_i} = \frac{2\cos\theta_i}{(n_2/n_1)\cos\theta_i + \cos\theta_t}$$

For $\mathbf{E}$ *perp to the plane of incidence*

$$r_\perp = \frac{E_r}{E_i} = \frac{\cos\theta_i - (n_2/n_1)\cos\theta_t}{\cos\theta_i + (n_2/n_1)\cos\theta_t} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$t_\perp = \frac{E_t}{E_i} = \frac{2\cos\theta_i}{\cos\theta_i + (n_2/n_1)\cos\theta_t}$$

If $n = n_2/n_1$:

$r_{\parallel} = 0$ at the Brewster angle, $\theta_i = \theta_B$, where $\tan\theta_B = n$

At normal incidence $(\theta_i = 0)$, reflecting power, $R_0 = r_{\parallel}^2 = r_{\perp}^2 = \left(\frac{n-1}{n+1}\right)^2$

$T_0 = \frac{Z_1}{Z_2} t^2 = n t^2 = \frac{4n}{(1+n)^2} \qquad (R+T) = 1$

EM waves in a conducting medium

Wave equation: $\frac{\partial^2 \mathbf{E}}{\partial x^2} = \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$

Solution: $E_y = E_0 \exp i(kx - \omega t)$ with $k^2 = i\omega\mu_0\sigma$ (k is complex)

Skin depth, $\delta = \sqrt{\left(\frac{2}{\mu_0 \sigma \omega}\right)}$ Effective (complex) refractive index $n = \frac{kc}{\omega}$

Reflection from metal at normal incidence $r = \frac{E_r}{E_i} = \left(\frac{1-n}{1+n}\right) \approx -1$

Transmission Lines
(here L and C are the inductance and capacitance per unit length)

Wave velocity, $v = \frac{1}{\sqrt{LC}}$ Characteristic impedance, $Z_0 = \sqrt{\frac{L}{C}}$

Twin wires, separation b, each of radius a, $C = \frac{\pi\epsilon_r\epsilon_0}{\ln(b/a)}$ $L = \frac{\mu_r\mu_0}{\pi}\ln(b/a)$

for air/vacuum: $Z_0 = 120\ln(b/a)\ \Omega$

Co-axial cable, radii a (inner) and b (outer), $C = \frac{2\pi\epsilon_r\epsilon_0}{\ln(b/a)}$ $L = \frac{\mu_r\mu_0}{2\pi}\ln(b/a)$

for air/vacuum: $Z_0 = 60\ln(b/a)\ \Omega$

Stripline: two conductors of width b, separation a, $C = \frac{\epsilon_r\epsilon_0 b}{a}$ $L = \frac{\mu_r\mu_0 a}{b}$

for air/vacuum: $Z_0 = 377\frac{a}{b}\ \Omega$

Waveguide equation $\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{(2b)^2}$

Lorentz transformation equations for **E** and **B**:

If O' is moving with speed $+V$ along x axis of O:

$$E'_x = E_x \qquad E'_y = \gamma(E_y - VB_z) \qquad E'_z = \gamma(E_z + VB_y)$$

$$B'_x = B_x \qquad B'_y = \gamma(B_y + \frac{V}{c^2}E_z) \qquad B'_z = \gamma(B_z - \frac{V}{c^2}E_y)$$

$B^2 - \frac{E^2}{c^2}$ and $\mathbf{E}\cdot\mathbf{B}$ are invariants.

Potentials:

Inside a solenoid of radius R, at distance a from the axis: $\mathbf{A} = \frac{\mu_0 n I a}{2}\hat{\phi}$

Outside the solenoid, $\mathbf{A} = \frac{\mu_0 n I R^2}{2a}\hat{\phi}$

$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla\times\mathbf{A}$.

Lorentz Gauge: $\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$

$$\left(\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \nabla^2 V\right) = \frac{\rho}{\epsilon_0} \qquad \left(\frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A}\right) = \mu_0\mathbf{J}$$

Liénard-Wiechert potentials (from a moving charge):

$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{qc}{(Rc - \mathbf{R}\cdot\mathbf{v})}$ and $\mathbf{A}(\mathbf{r},t) = \frac{\mathbf{v}}{c^2}V(\mathbf{r},t)$ where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$

Electric Dipole Radiation:

$$V(\mathbf{r},t) = \frac{-p_0\,\omega\cos\theta}{4\pi\epsilon_0 rc}\sin\omega t' \qquad \mathbf{A}(\mathbf{r},t) = \frac{-\mu_0\,p_0\,\omega}{4\pi r}\sin\omega t'\ \hat{\mathbf{z}}$$

Magnetic Dipole Radiation:

$$V(\mathbf{r},t) = 0 \qquad \mathbf{A}(\mathbf{r},t) = \frac{-\mu_0\,m_0\,\omega\sin\theta}{4\pi rc}\sin\omega t'\ \hat{\phi}$$

NB $t' = t - |\mathbf{r} - \mathbf{r}'|/c$ (Retarded time.)

In both cases, **E** is $\perp$ to **B**, and both are $\perp$ to **r**, and $E = cB$.