Examination 2005
Quantum Mechanics PHYS3210
Time: 2 hours
Total number of questions 3. Answer all questions.
The questions are of equal value.
Calculators may be used.
This paper may be retained by the candidate.

$$\hbar = 1.054 \times 10^{-34} Js$$

$$1eV = 1.602 \times 10^{-19} J \quad .$$

Question 1

a) The spherical harmonics $|l, m\rangle = Y_{lm}$ are simultaneous eigenfunctions of the operators $\hat{l}^2 = \hat{L}^2/\hbar^2$ and $\hat{l}_z = \hat{L}_z/\hbar$.

(i) (1mark) Write down eigenvalue equations for the two operators, showing clearly the possible values of l and m.

(ii) (1mark) Because of rotational symmetry

$$\langle \hat{l}_x^2 \rangle = \langle \hat{l}_y^2 \rangle ,$$

where $\langle ... \rangle = \langle lm | ... | lm \rangle$. Using the explicit form of $\hat{l}^2$, $\hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2$, prove that

$$\langle \hat{l}_x^2 \rangle = \frac{1}{2}[l(l+1) - m^2] .$$

(iii) (2marks) Using the raising and lowering operators $\hat{l}_\pm = \hat{l}_x \pm i\hat{l}_y$ show that

$$\langle \hat{l}_x \rangle = \langle \hat{l}_y \rangle = 0 .$$

(Remember that $\hat{l}_\pm$ acting on $|lm\rangle$ always changes value of m.)

b) The first three rotational energy levels of the SiO molecule are

$$\begin{aligned} \epsilon_0 &= 0, \\ \epsilon_1 &= 1.8 \times 10^{-4} eV, \\ \epsilon_2 &= 5.4 \times 10^{-4} eV. \end{aligned}$$

The molecule consists of isotopes ^{28}Si and ^{16}O, $M_O = 26.5 \times 10^{-27} kg$, $M_{Si} = 46.5 \times 10^{-27} kg$.

(i) (2marks) What is the degeneracy of each level? Write wave functions of all quantum states corresponding to these levels in terms of spherical harmonics Y_{lm}.

(ii) (4marks) Using the energy levels find the moment of inertia I of the molecule and the distance between the Silicon and the Oxygen nuclei.

Question 2

a) A quantum system has Hamiltonian $\hat{H} = \hat{H}_0 + \delta\hat{V}$, where $\hat{H}_0$ is independent of time and $\delta\hat{V}$ is a small time-dependent perturbation. At $t = \pm\infty$ the perturbation vanishes, $\delta\hat{V}(t = \pm\infty) = 0$. The eigenstates and eigenvalues of $\hat{H}_0$ are known. At $t = -\infty$ the system is in some eigenstate ψ_i. The wave function at time t can be written as

$$\psi(r,t) = \sum_n c_n(t) e^{-i\epsilon_n t/\hbar} \psi_n(r) \ .$$

(i) (1mark) Explain the physical significance of the coefficients $c_n(t)$ in this expression. What are values of c_n at $t = -\infty$?
(ii) (3marks) Starting from $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$ derive a set of differential equations for $c_n(t)$.
(iii) (2marks) Show that to the leading order in $\delta\hat{V}$ the probability of transition from state $|i>$ to state $|f>$ is

$$W_{if} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{+\infty} < \psi_f | \delta\hat{V}(t) | \psi_i > e^{(\epsilon_f - \epsilon_i)t/\hbar} dt \right|^2 \ .$$

b) Consider a hydrogen atom in the ground state, $\psi_{1s}(r) = \frac{1}{\sqrt{\pi a_B^3}} e^{-r/a_B}$. A short pulse of electric field $\mathcal{E}$ is applied to the atom. The pulse is so short that the time dependence can be considered as a δ-function.

$$\delta\hat{V} = -\tau e \mathcal{E} z \delta(t) = -\tau e \mathcal{E} r \cos\theta \delta(t) \ .$$

Here e is the electric charge of electron and τ is some characteristic time.
(i) (3marks) Using time-dependent perturbation theory calculate the probability of excitation of the atom to the $2p_0$-state

$$\psi_{2p_0} = \sqrt{\frac{3}{4\pi}} \cos\theta \frac{1}{2\sqrt{6a_B^3}} \frac{r}{a_B} e^{-r/(2a_B)} \ .$$

You can use without proof the following standard integral, $\int_0^\infty y^n e^{-y} dy = n!$
(ii) (1mark) What is value of the excitation probability for $\mathcal{E} = 10^9 V/m$ and $\tau = 10^{-16} sec$?

Question 3

The spin operator $\hat{\vec{s}}$ for a particle with spin 1/2 (for example an electron) can be expressed in terms of Pauli matrixes $\hat{\vec{\sigma}}$, $\hat{\vec{s}} = \frac{1}{2}\hat{\vec{\sigma}}$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

a) (4marks) Calculate $\hat{s}^2 = \hat{s}_x^2 + \hat{s}_y^2 + \hat{s}_z^2$ and comment on the result.

b) (6marks) An electron is placed in a static magnetic field $\vec{B} = B(\sin\theta, 0, \cos\theta)$. The interaction of the electron spin with the magnetic field is given by the following Hamiltonian

$$\hat{H} = -\frac{e\hbar}{m_e}(\vec{B} \cdot \hat{\vec{s}}) = -\frac{e\hbar}{2m_e} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} . \qquad (1)$$

Calculate the eigenvalues and corresponding eigenvectors and discuss the physical significance of your results.