THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

PHYS2050 – Electromagnetism

Session 2, 2010

1. Time allowed – 2 hours
2. Total number of questions – 5
3. Total marks available – 60
4. Answer ALL questions
5. ALL QUESTIONS ARE NOT OF EQUAL VALUE.
   Marks available for each question are shown in the examination paper.
6. Answer Questions 1–2 alone in one book.
7. Answer Questions 3–5 in a separate book.
8. University-approved calculators may be used.
9. All answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.
10. This paper may be retained by the candidate.

PHYS2050 — Definitions and Formulae

Gradient

$$\nabla f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$$

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$$

Laplacian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\,\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Divergence

$$\nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\,A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$\nabla\cdot\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Curl

$$\nabla\times\boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}$$

$$\nabla\times\boldsymbol{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\,A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\phi}$$

$$\nabla\times\boldsymbol{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]\hat{z}$$

Identities

$$\nabla\times(\nabla f) = 0$$
$$\nabla\cdot(\nabla\times\boldsymbol{A}) = 0$$
$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

Gradient Theorem

$$\int_{\boldsymbol{a}}^{\boldsymbol{b}}(\nabla f)\cdot d\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a})$$

Divergence Theorem

$$\oint_S \boldsymbol{A}\cdot d\boldsymbol{a} = \int_V \nabla\cdot\boldsymbol{A}\,dv$$

Stokes' Theorem

$$\oint_L \boldsymbol{A}\cdot d\boldsymbol{l} = \int_S(\nabla\times\boldsymbol{A})\cdot d\boldsymbol{a}$$

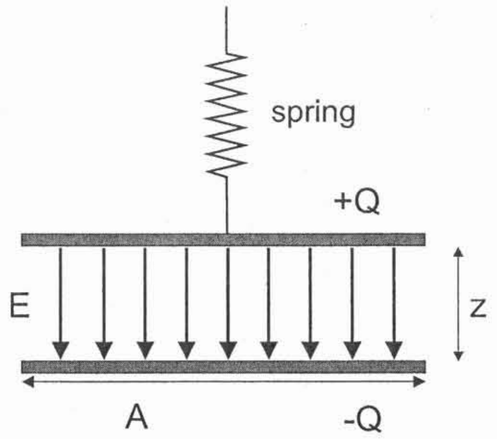
Coulomb's Law	$F = \frac{q_1 q_2}{4\pi\epsilon_0 r^2}$
Biot-Savart Law	$\boldsymbol{dB} = \frac{\mu_0}{4\pi}\frac{I\boldsymbol{dl} \times \hat{r}}{r^2}$
Lorentz Force Law	$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$
Maxwell's Equations	$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$
	$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}\right)$
Electric Potential	$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}) - \frac{\partial \boldsymbol{A}}{\partial t}$
Vector Potential	$\boldsymbol{B} = \nabla \times \boldsymbol{A}\,, \qquad \boldsymbol{A} = \frac{\mu_0}{4\pi}\int_V \frac{\boldsymbol{J}}{r} dv$
Charge Conservation	$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$
Current Density	$\boldsymbol{J} = \rho\,\boldsymbol{v} = \rho\mu\,\boldsymbol{E} = \sigma\boldsymbol{E}$
Poynting Vector	$\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E} \times \boldsymbol{B}$
Stored Energy	$W = \frac{1}{2}\sum_i q_i V(\boldsymbol{r}_i) = \frac{1}{2}\int V(\boldsymbol{r})\rho(\boldsymbol{r})\, d^3\boldsymbol{r}$
	$W = \frac{1}{2}\int \left(\epsilon E^2 dv + \frac{B^2}{\mu}\right) dv \qquad W = \frac{1}{2}\int \boldsymbol{D} \cdot \boldsymbol{E}\, dv$
Faraday's Law	$\int \boldsymbol{E} \cdot \boldsymbol{dl} = -\frac{d\phi}{dt} \qquad \phi = \int_S \boldsymbol{B} \cdot \boldsymbol{da}$
Electric Displacement	$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \qquad \int_S \boldsymbol{D} \cdot \boldsymbol{ds} = Q_f$
Magnetic Field Intensity	$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \qquad \int_L \boldsymbol{H} \times \boldsymbol{dl} = I_f$
Bound Charge Densities	$\rho_b = -\nabla \cdot \boldsymbol{P} \qquad \sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$
Magnetization Current Densities	$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} \qquad \boldsymbol{K}_b = \boldsymbol{M} \times \hat{\boldsymbol{n}}$
Linear Media	$\boldsymbol{P} = \chi_e\epsilon_0\boldsymbol{E} \qquad \boldsymbol{D} = \epsilon_0(1 + \chi_e)\boldsymbol{E} = \epsilon\boldsymbol{E}$
	$\boldsymbol{M} = \chi_m\boldsymbol{H} \qquad \boldsymbol{B} = \mu_0(1 + \chi_m)\boldsymbol{H} = \mu\boldsymbol{H}$
Capacitance	$C = \frac{Q}{V} \qquad W = \frac{1}{2}CV^2$
Inductance	$\varepsilon = -L\frac{dI}{dt} \qquad W = \frac{1}{2}LI^2$

Physical Constants

Avagadro's Number	N_A	6.02×10^{23} mole^{-1}
Permittivity of free space	ϵ_0	8.85×10^{-12} Fm^{-1}
	$\frac{1}{4\pi\epsilon_0}$	9.0×10^{9} Nm^{2}C^{-2}
Permeability of free space	μ_0	$4\pi \times 10^{-7}$ Hm^{-1}
Electron charge	e	1.6×10^{-19} C
Electron mass	m_e	9.11×10^{-31} kg
Proton mass	m_p	1.67×10^{-27} kg
Speed of light in vacuum	c	3.00×10^{8} ms^{-1}

Question 1 (10 marks)
Force between two capacitor plates.

Two plates of a capacitor have an area of A and are separated by a distance z. They have a charge of $+Q$ and $-Q$, respectively. The force between the capacitor plates is measured with a spring. The stray field at both sides of the capacitor should be neglected.



(a) Show that the force between the capacitor plates is $F = -\frac{1}{2}\,E\,Q$.

(b) Determine the voltage between the capacitor plates.

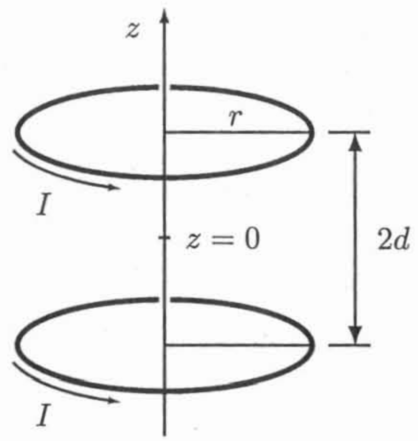
(c) How does the force change if the space between both plates is filled with an insulating liquid material with dielectric constant ε_r?

Question 2 (10 marks)

A wonderfully uniform magnetic field can be obtained inside an infinitely long solenoid. Unfortunately these are a little impractical in the laboratory. So instead we use a "**Helmholtz coil**" – an arrangement of two circular wires of radius r carrying equal currents as shown in the figure.

(a) Find the field $\boldsymbol{B}$ along the axis of the coils as function of z.

(b) Show that the derivative $\partial B/\partial z$ vanishes halfway between the coils, and hence that the field is uniform here.



Question 3 (12 marks)

(a) Starting from Maxwell's equations, calculate the magnetic field $\boldsymbol{B}$ due to a infinite straight wire lying along the z-axis and carrying a current I.

(b) What is the direction of the vector potential $\boldsymbol{A}$?
Hint: you might like to use the formula

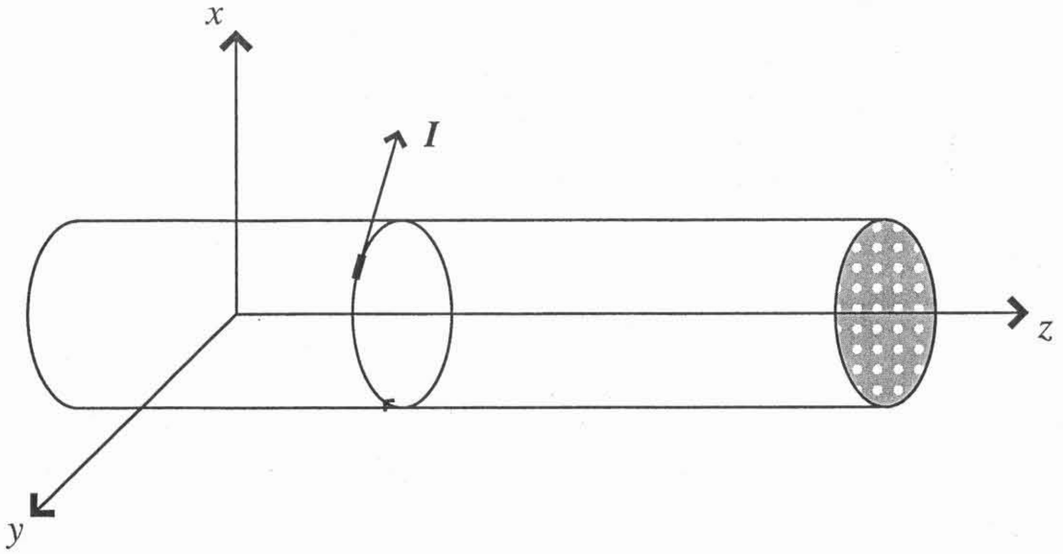
$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{I}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} dl'$$

(c) Calculate $\boldsymbol{A}$ using any method you like. Explain where you have set $\boldsymbol{A} = 0$.

(d) Prove that your answer is correct by explicitly showing that
$\nabla \times \boldsymbol{A} = \boldsymbol{B}$ and $\nabla \cdot \boldsymbol{A} = 0$.

Question 4 (14 marks)

A long solenoid with a circular cross-section of radius r and a total of N turns along its length l carries a current I as shown in the diagram. N is large enough that the current in the direction of $\hat{z}$ (along the axis) can be neglected, and l is large enough that you can neglect edge effects.



(a) Calculate the magnetic field $\boldsymbol{B}$ everywhere.

(b) Find the vector potential $\boldsymbol{A}$ everywhere.

(c) Find the (self) inductance of the solenoid.

(d) Assume the solenoid is initially carrying no current, when at time $t = 0$ it is suddenly connected in series to a 9V battery and a resistor R. Using Ohm's law, find an expression for the current as a function of time. Sketch $I(t)$.

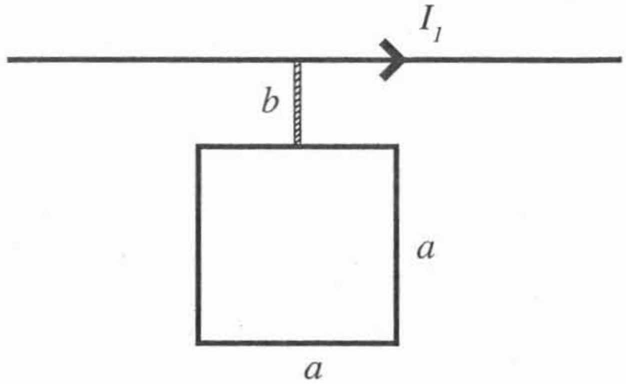
(e) Calculate the power output of the battery as a function of time.

(f) Calculate the rate at which energy is dissipated by the resistor as a function of time.

(g) Integrate the difference between parts (e) and (f) from time $t = 0$ to ∞ in order to get the total energy difference. Where is this energy? Is its value what you expect?

Question 5 (14 marks)

A light, square loop of side length a hangs below a very long, straight wire carrying a time-varying current $I_1(t)$. The distance between the long wire and the top of the loop is b. Assume that the loop is rigid and cannot twist (neglect torsion).



(a) Calculate the magnetic flux through the loop resulting from the current I_1.

(b) Show that the electromotive force induced in the loop is given by

$$\varepsilon = \frac{\mu_0 a}{2\pi} \ln \frac{a+b}{b} \frac{dI_1(t)}{dt} .$$

(c) Assume that the loop is made of conducting material (conductance σ) and has a cross-sectional area A. Find the current in the loop and indicate its direction.

(d) Find the net Lorentz force on the loop.

(e) If the current in the wire is increasing linearly with time, $I_1 = kt$, where k is constant, find the force as a function of time.

(f) When the force becomes large enough, the string will break. Assuming the string can withstand a tension T, find the time at the point that the string breaks.