115

THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

Session 2, 2009

PHYS2050 – Electromagnetism

2009

1. Time allowed – 2 hours
2. Total number of questions – 5
3. Total marks available – 50
4. Answer ALL questions
5. ALL QUESTIONS ARE OF EQUAL VALUE.
   Marks available for each question are shown in the examination paper.
6. Answer Question 1 alone in one book.
7. Answer Questions 2–5 in a separate book.
8. University-approved calculators may be used.
9. All answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.
10. This paper may be retained by the candidate.

PHYS2050 Definitions and Formulae

Gradient	$\nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$
Divergence	$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
Curl	$\nabla \times \mathbf{A} = \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\hat{\mathbf{i}} + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\hat{\mathbf{j}} + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\hat{\mathbf{k}}$
Laplacian	$\nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
Identities	$\nabla \times (\nabla f) = 0 \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0$ $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$
Gradient Theorem	$\int_a^b (\nabla f) \cdot \mathbf{dl} = f(b) - f(a)$
Divergence Theorem	$\int_S \mathbf{A} \cdot \mathbf{ds} = \int_V (\nabla \cdot \mathbf{A}) dv$
Stokes' Theorem	$\int_L \mathbf{A} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{ds}$
Coulomb's Law	$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2}$
Electric Field	$\mathbf{F} = Q\mathbf{E} \qquad \mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_v}{r^2}\hat{\mathbf{r}} dv$
Gauss' Law	$\int_S \mathbf{E} \cdot \mathbf{ds} = \frac{Q}{\epsilon_0} \qquad \nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon_0}$
Electric Potential	$V(b) - V(a) = -\int_a^b \mathbf{E} \cdot \mathbf{dl} \qquad \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$ $\nabla^2 V = -\frac{\rho_v}{\epsilon_0} \qquad V = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_v dv}{r}$
Current Density	$\mathbf{J} = \rho_v \mathbf{v} = \rho_v \mu \mathbf{E} = \sigma \mathbf{E}$
Charge Conservation	$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$
Stored Energy	$W = \frac{1}{2}\sum q_i V(\mathbf{r}_i) \qquad W = \frac{1}{2}\int_V V(\mathbf{r})\rho(\mathbf{r}) dv = \frac{1}{2}\int_V \epsilon E^2 dv$ $W = \frac{1}{2}\int \mathbf{D} \cdot \mathbf{E} d\tau$
Bound Charge Densities	$\rho_{vb} = -\nabla \cdot \mathbf{P} \qquad \sigma_{sb} = \mathbf{P} \cdot \hat{\mathbf{n}}$
Electric Displacement	$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad \int_S \mathbf{D} \cdot \mathbf{ds} = Q_f$
Linear Media	$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \qquad \mathbf{D} = \epsilon_0(1 + \chi_e)\mathbf{E} = \epsilon \mathbf{E}$ $\epsilon = K\epsilon_0 = (1 + \chi_e)\epsilon_0$
Boundary Conditions	$D_{n1} = D_{n2} \qquad E_{t1} = E_{t2}$
Capacitance	$C = \frac{Q}{V}$
Biot-Savart Law	$\mathbf{dB} = \frac{\mu_0 I}{4\pi} \frac{\mathbf{dl} \times \hat{\mathbf{r}}}{r^2}$

Magnetic Induction	$\int_L \mathbf{B}\cdot d\mathbf{l} = \mu_0 I$	$\int_S \mathbf{B}\cdot d\mathbf{s} = 0$
Solenoid	$B = \mu_0 n I$	
Straight Wire	$B = \frac{\mu_0 I}{2\pi r}$	
Vector Potential	$\mathbf{B} = \nabla \times \mathbf{A}$	$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{J}dv}{r}$
Magnetic Forces	$\mathbf{F} = Q\mathbf{v}\times\mathbf{B}$	$d\mathbf{F} = I d\mathbf{l}\times\mathbf{B}$
Magnetization Current Densities	$\mathbf{J}_b = \nabla\times\mathbf{M}$	$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$
Magnetic Field Intensity	$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$	$\int_L \mathbf{H}\times d\mathbf{l} = I_f$
Linear Media	$\mathbf{M} = \chi_m \mathbf{H}$	$\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \mu\mathbf{H}$
	$\mu = \mu_0(1+\chi_m) = \mu_0\mu_r$	
Boundary Conditions	$H_{t1} = H_{t2}$	$B_{n1} = B_{n2}$
Stored Energy	$W = \frac{1}{2}\int \frac{B^2}{\mu} dv$	
Faraday's Law	$\int \mathbf{E}\cdot d\mathbf{l} = -\frac{d\phi}{dt}$	$\phi = \int_S \mathbf{B}\cdot d\mathbf{s}$
Polarization Current	$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$	
Maxwell's Equations	$\nabla\cdot\mathbf{E} = \frac{\rho_v}{\epsilon_0}$	$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$
	$\nabla\times\mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0\frac{\partial\mathbf{E}}{\partial t})$	$\nabla\cdot\mathbf{B} = 0$
	$\int_L \mathbf{B}\cdot d\mathbf{l} = \mu_0(I + I_d) = \mu_0 \mathbf{I} + \mu_0\epsilon_0\frac{\partial\phi_E}{\partial t}$	
	$\nabla\cdot\mathbf{D} = \rho_{vf}$	$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$
	$\rho_v = \rho_{vf} + \rho_{vb}$	$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_m + \mathbf{J}_p$
Poynting Vector	$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$	
Electromagnetic Waves	$v = \frac{1}{\sqrt{\epsilon\mu}}$	$\eta = \sqrt{\frac{\mu}{\epsilon}}$

Physical Constants

Permittivity of free space	ϵ_0	8.85×10^{-12} F/m
	$\frac{1}{4\pi\epsilon_0}$	$9.0\times10^{9}\ Nm^2/C^2$
Permeability of free space	μ_0	$4\pi\times10^{-7}$ H/m
Electron charge	e	1.6×10^{-19} C
Speed of light in vacuum	c	3.00×10^{8} m/s

Question 1 (10 marks)

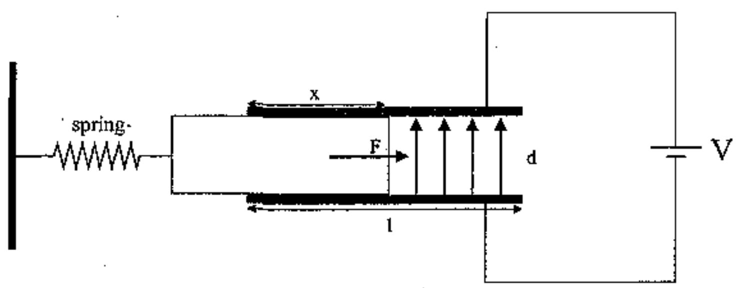
Force on a dielectric material in a capacitor.

Calculate the force which acts on a dielectric material (dielectric constant ε) which is partly (i.e. with a length of x) inside a capacitor. The area of the capacitor is $a \cdot l$ and the distance between the plates is d. Both plates are charged and constant potential V is maintained between them.

(a) Calculate the change in the stored energy of the capacitor when shifting the dielectric material by dx.

(b) Calculate the work done by the battery when the dielectric material is shifted by dx.

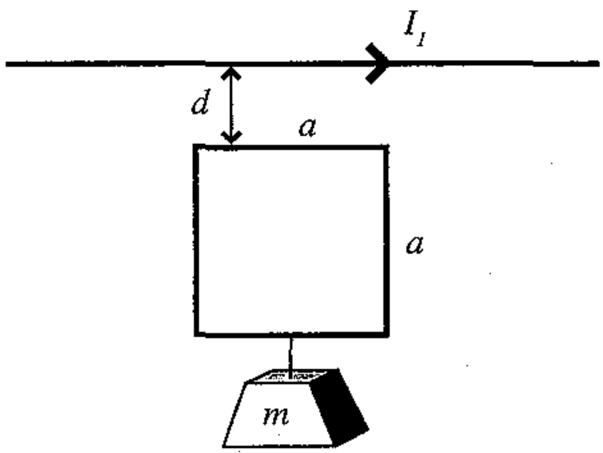
(c) Calculate the force which acts on the spring on the other side of the dielectric material.



Question 2 (10 marks)

(a) Two infinite, parallel wires are separated by a distance d. One carries a current I_1 and the other I_2. Using Maxwell's equations and the Lorentz force law, derive an expression for the force per unit length between the wires.

(b) A square loop of side length a hangs below a very long, straight wire carrying a current I_1. The distance between the long wire and the top of the loop is d. Attached to the loop is a mass m.

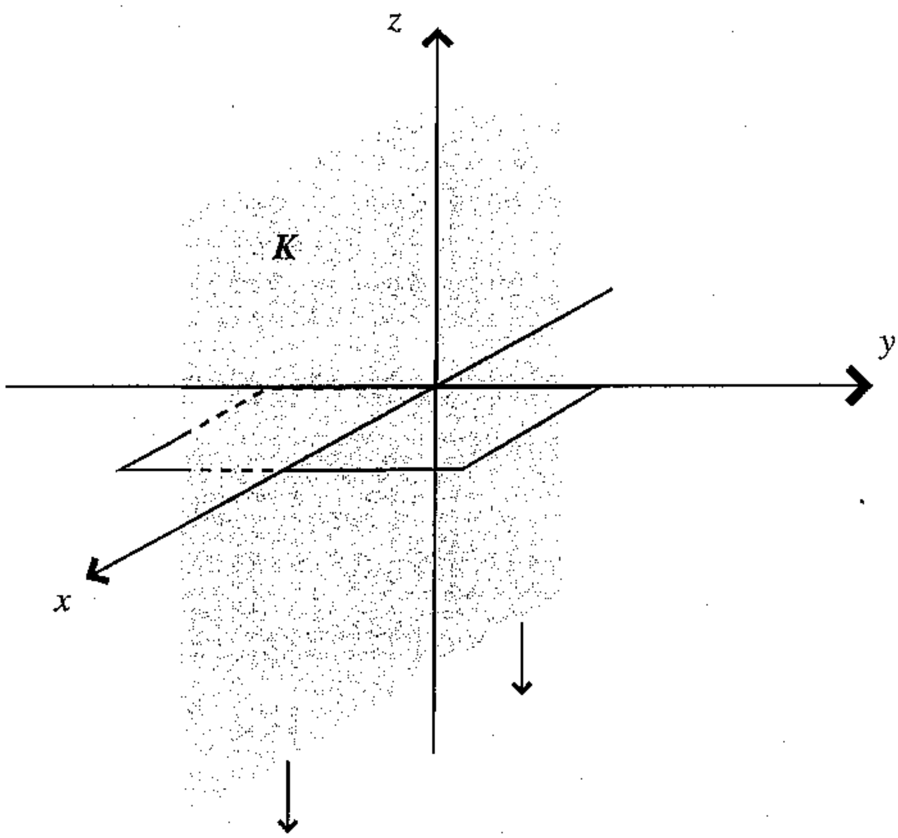


i. For what current I_2 in the loop will the magnetic force upward balance the gravitational force downward? Should it flow clockwise or anticlockwise?

ii. What is the total magnetic force on the long wire due to the current in the loop?

Question 3 (10 marks)

An infinite uniform surface current $\mathbf{K} = -K\hat{z}$ flows over the xz plane (in the negative $\hat{z}$ direction).



(a) What is the direction of the magnetic field $\mathbf{B}$ both to the left and right of the surface? Justify your answer.

(b) Apply Ampere's law to obtain the field $\mathbf{B}$. (Hint: use a rectangular amperian loop that runs parallel to the xy plane and extends an equal distance either side of the surface current, see diagram.)

(c) What is the direction of the vector potential $\mathbf{A}$? (Hint: use the formula

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}da'$$

and note that $\mathbf{K}$ is always in the same direction.)

(d) Calculate $\mathbf{A}$ in the region $y > 0$. Note that the formula presented above should not be used to calculate $\mathbf{A}$ in this case because $\mathbf{K}$ does not go to zero at infinity. Rather, use the formula

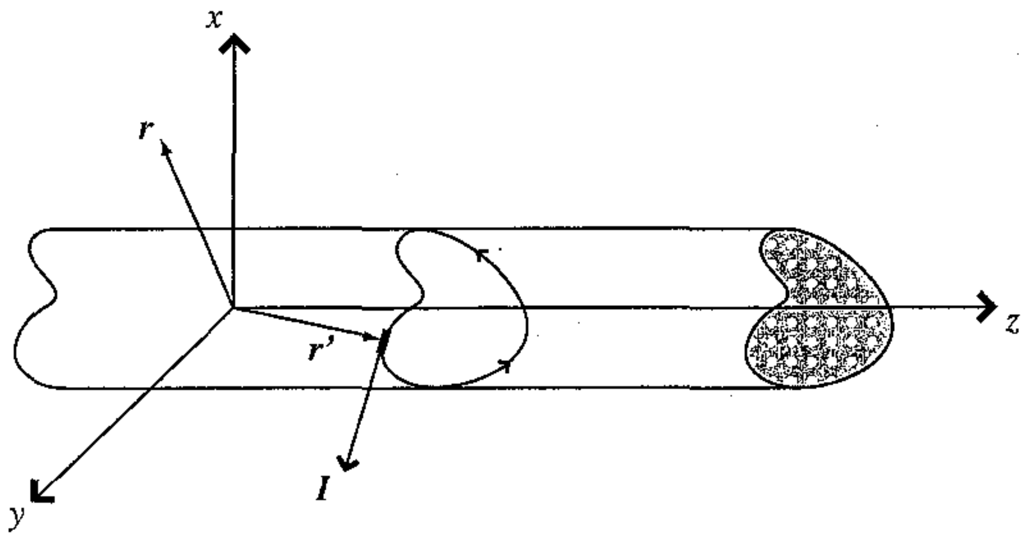
$$\nabla \times \mathbf{A} = \mathbf{B}$$

and Stokes' theorem to determine $\mathbf{A}$. Set the vector potential to zero at the current-carrying surface.

(e) Prove that your answer to Part (d) is correct by explicitly showing that $\nabla \times \mathbf{A} = \mathbf{B}$.

Question 4 (10 marks)

In this question you will show that the magnetic field of an infinite solenoid runs parallel to its axis regardless of the cross-sectional shape of the solenoid (as long as the shape is constant throughout the solenoid). The solenoid has n turns per unit length carrying a current I, where n is a very large number.



Consider the field at $\mathbf{r} = (x, y, 0)$ due to the current element at $\mathbf{r}' = (x', y', z')$.

(a) Write an expression for the current $\mathbf{I}$ at $\mathbf{r}'$ in Cartesian coordinates. Remember that the current has no component along the solenoid.

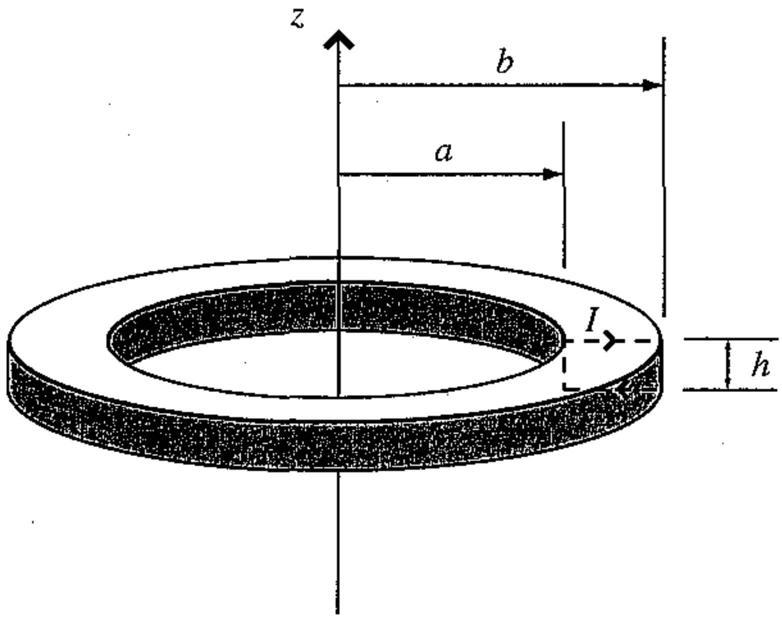
(b) Use the Biot-Savart law to obtain the field element $\mathbf{dB}$ at $\mathbf{r}$ due to the current at $\mathbf{r}'$.

(c) By noting that there is a symmetrically situated current element at $\mathbf{r}''$ that has the same x and y as $\mathbf{r}'$ but *negative* z, show that the field at $\mathbf{r}$ is in the $\hat{z}$ direction, and hence that the field in general is axial.

(d) Now that you have its direction, use Ampére's law (or any other method you like) to calculate the field everywhere.

Question 5 (10 marks)

Consider a superconducting toroidal coil with a rectangular cross-section. Its inner radius is a, outer radius is b, and height is h. The coil has a total of N turns that carry a current I, where N is large enough that the current in the direction of $\hat{\phi}$ (around the axis) can be neglected.



(a) Calculate the magnetic field $\mathbf{B}$ everywhere (you may assume the direction of the field is $\hat{\phi}$).

(b) Calculate the magnetic energy stored in the coil.

(c) Find the (self) inductance of the coil.

(d) The coil is carrying a current I_0 when it is slowly warmed up, until at time $t = 0$ it reaches a critical temperature where the coil is no longer superconducting. It now has a total resistance R and therefore the current begins to change. What is the induced emf in the circuit?

(e) Using Ohm's law, find an expression for the current I as a function of time. Sketch $I(t)$.

(f) What is the power dissipated by the coil as a function of time?

(g) Integrate the power over time to get the total energy dissipated by the coil when it stops being a superconductor. Compare your answer with Part (b). Is the answer what you expect?