THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

NOVEMBER 2008

PHYS2050

Electromagnetism

Time Allowed – 2 hours

Total number of questions - 5

Answer ALL questions

Answer each question in a separate book

All questions ARE of equal value

Candidates may not bring their own calculators.

The following materials will be provided by the Enrolment and Assesment Section: Calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

p3

CARTESIAN: $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $d\boldsymbol{\tau} = dx\,dy\,dz$

Gradient $$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl $$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Laplacian $$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

SPHERICAL: $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}$; $d\boldsymbol{\tau} = dl_r dl_\theta dl_\phi = r^2 \sin\theta\,dr\,d\theta\,d\phi$

Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian $$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

CYLINDRICAL: $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $d\boldsymbol{\tau} = r\,dr\,d\phi\,dz$

Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl $$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right)\hat{\mathbf{z}}$$

Laplacian $$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f\nabla g + g\nabla f$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla\cdot(\nabla\times\mathbf{v}) = 0$

(10) $\nabla\times(\nabla T) = 0$

(11) $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

FUNDAMENTAL THEOREMS

Gradient Theorem $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$

Divergence Theorem $\int_{volume} (\nabla\cdot\mathbf{v})d\mathbf{r} = \oint_{surface} \mathbf{v}\cdot d\mathbf{a}$

Curl Theorem $\int_{surface} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint_{boundary\ line} \mathbf{v}\cdot d\mathbf{l}$

Coulomb's Law $\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}}$

Electric Field $\mathbf{F} = Q\mathbf{E}$ $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(r)}{r^2}\hat{\mathbf{r}}d\mathbf{r}$

Gauss' Law $\int \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{enclosed}}{\varepsilon_0}$ $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$

Electric Potential $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{E}\cdot d\mathbf{l}$ $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$

Poisson's Eqn $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ $V = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r)}{r}d\mathbf{r}$

Laplace's Eqn:- $\nabla^2 V = 0$ Sphere $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

Cylinder $V(r,\phi) = a_0\ln r + c_0 + \sum_{k=1}^{\infty}\left(c_k r^k + d_k r^{-k}\right)(a_k\cos k\phi + b_k\sin k\phi)$

Boundary Conditions $V_{above} = V_{below}$ $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

Legendre polynomials $P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Stored energy $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$ $W = \frac{1}{2}\int \rho(\mathbf{r})V(\mathbf{r})d\mathbf{r} = \frac{\varepsilon_0}{2}\int E^2 d\mathbf{r}$

Bound charge densities $\rho_b = -\nabla\cdot\mathbf{P}$ $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$

Electric Displacement $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f\ \text{enclosed}}$

Linear Dielectrics $\mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$ $\mathbf{D} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon\mathbf{E}$

$\varepsilon = \varepsilon_r\varepsilon_0 = (1+\chi_e)\varepsilon_0$

Boundary Conditions $\mathbf{D}^{\parallel}_{above} - \mathbf{D}^{\parallel}_{below} = \mathbf{P}^{\parallel}_{above} - \mathbf{P}^{\parallel}_{below}$ $D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$

Capacitance $C = \frac{Q}{V}$

Biot-Savart	$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}' \times \hat{\mathbf{s}}}{s^2}$	
Ampere's Law	$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc}$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$
Vector Potential	$\mathbf{B} = \nabla \times \mathbf{A}$	$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'}{s}$
Magnetic Forces	$\mathbf{F} = q\mathbf{v}\times\mathbf{B}$	$d\mathbf{F} = I d\mathbf{l}\times\mathbf{B}$
Bound current densities	$\mathbf{J}_b = \nabla \times \mathbf{M}$	$\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$
Boundary Conditions	$\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}})$	
	$\mathbf{A}_{above} = \mathbf{A}_{below}$	$\frac{\partial}{\partial n}\mathbf{A}_{above} - \frac{\partial}{\partial n}\mathbf{A}_{below} = -\mu_0\mathbf{K}$
Linear Media	$\mathbf{M} = \chi_m \mathbf{H}$	$\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \mu\mathbf{H}$
	$\mu = \mu_r\mu_0 = (1+\chi_m)\mu_0$	
Stored Energy	$W = \frac{1}{2\mu}\int B^2 d\mathbf{r}$	

Faraday's Law $\int \mathbf{E}\cdot d\mathbf{l} = -\frac{d\phi}{dt}$ $\phi = \int_S \mathbf{B}\cdot d\mathbf{a}$

Inductance $M_{21} = \frac{\mu_0}{4\pi}\iint \frac{d\mathbf{l}_1\cdot d\mathbf{l}_2}{s}$ $\varepsilon = -L\frac{dI}{dt}$

Maxwell's Equations

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)$$

Polarization Current $\mathbf{J}_p = \frac{\partial\mathbf{P}}{\partial t}$

Maxwell's Equations in Matter

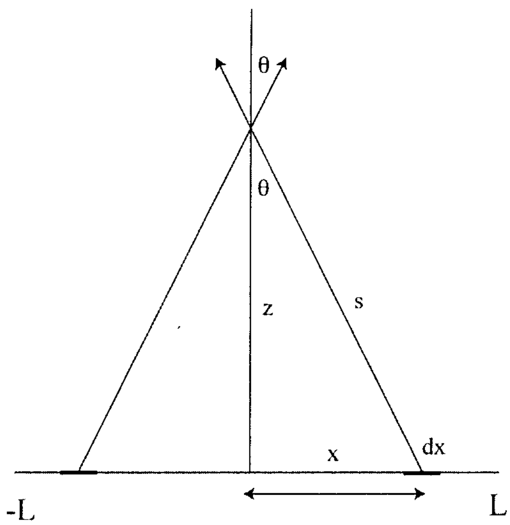
$$\nabla\cdot\mathbf{D} = \rho_f$$
$$\nabla\cdot\mathbf{B} = 0$$
$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\times\mathbf{H} = \mathbf{J}_f + \frac{\partial\mathbf{D}}{\partial t}$$

for linear media $\left.\begin{array}{l}\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}\\ \mathbf{M} = \chi_m\mathbf{H}\end{array}\right\}$ so that $\left.\begin{array}{l}\mathbf{D} = \varepsilon\mathbf{E}\\ \mathbf{H} = \frac{1}{\mu}\mathbf{B}\end{array}\right\}$

QUESTION 1 (20 marks)

a) Show that the electric field a distance z above the midpoint of a straight line segment of length $2L$, which carries a uniform line charge density λ is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{2\lambda L\,\hat{\mathbf{z}}}{z\sqrt{z^2 + L^2}}.$$



b) Explain in physical and mathematical terms the limits $z >> L$ and $L \rightarrow \infty$.

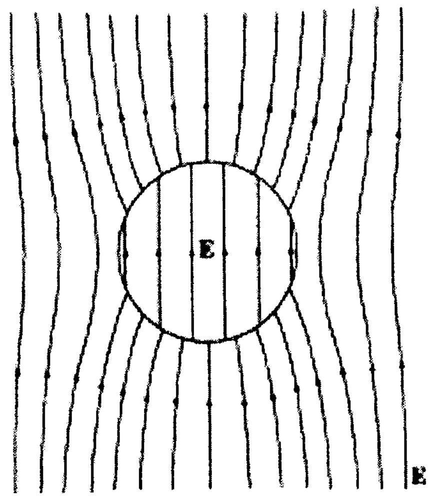
c) Find the potential a distance s from an infinitely long straight wire with a uniform line charge λ. Use $s = a$ as the reference point.

d) Find the energy of a uniformly charged spherical shell of total charge q, from the electric field

$$W = \frac{\varepsilon_0}{2} \int_{all\ space} E^2 d\mathbf{r}.$$

QUESTION 2 (20 marks)

A sphere of homogeneous linear dielectric material of dielectric constant ε is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$.



a) Write down the three boundary conditions the potential must satisfy.

b) Using the solution of Laplace's equation for spherical polar coordinates with symmetry in the ϕ direction,

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

show that the potential inside the sphere is given by

$$V(r,\theta) = -\frac{3E_0}{\varepsilon+2}z.$$

Hint: You can assume that the coefficients $A_l = B_l = 0$ for $l \neq 1$.

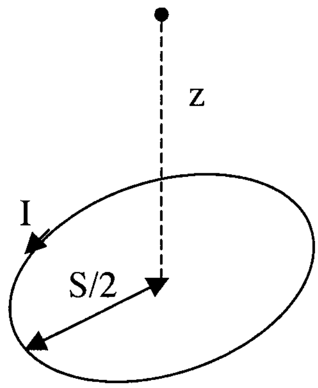
c) Calculate the electric field inside the sphere.

Question 3 (20 marks)

Part A

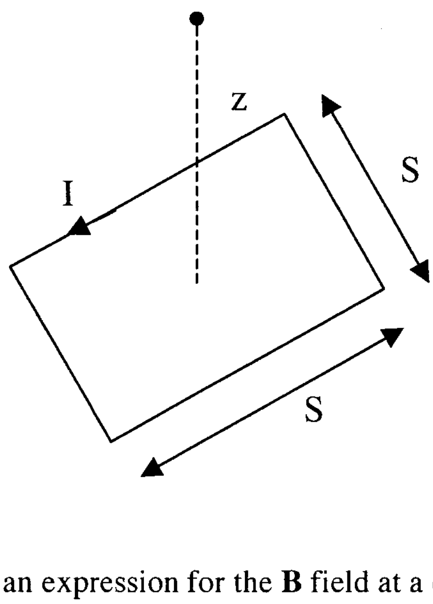
A current I flows through a circular wire loop of diameter S.

(i) Derive an expression for the **B** field (magnetic induction) at a distance Z above the centre of the loop and perpendicular to the loop.



Part B

An identical current I flows around a square loop of side S.



(ii) Derive an expression for the **B** field at a distance Z above the centre of the loop and perpendicular to the loop.

[Hint: $\int \frac{A\,dx}{\left(x^2 + A\right)^{3/2}} = \frac{x}{\left(x^2 + A\right)^{1/2}}$]

Part C

(iii) How would you expect these two fields to differ as Z becomes very large with respect to S?

(iv) Give an expression for **B** at large Z (if possible) and in simple terms, explain the source of the difference in **B** fields produced by the two loops.

Question 4 (20 marks)

Part A

(i) Show that for any vector field **W**, the divergence of the curl of the field is always equal to zero (i.e. $\nabla \cdot (\nabla \times \mathbf{W}) = 0$).

(ii) Using Maxwell's equations for electrodynamics (or otherwise), show that this theorem holds true for the electric field **E** (i.e. prove that $\nabla \cdot (\nabla \times \mathbf{E}) = 0$)

Part B

Ampere's original law can be written as:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

(iii) Show that this original form of Ampere's law is not consistent with vector calculus.

[Hint: take the divergence of the curl of **B** and show that it is not necessarily zero].

So as to be consistent with the theorems of vector calculus, Maxwell amended Ampere's law to:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

(iv) Show that this expression for Curl **B** satisfies the theorem that the divergence of the curl of a vector field is always zero.

The additional term $\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ is called the displacement current.

(v) Is it a current?
(vi) Explain its meaning and significance.
(vii) Give an example of how this term is useful.

Question 5 (20 marks)

Part A

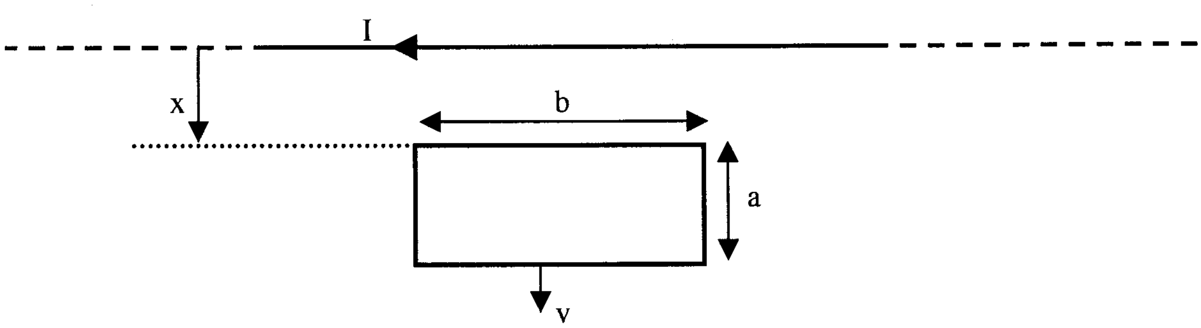
The wave equation for the electric field in free space is given by:

$$\nabla^2\mathbf{E} = \mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

(i) Derive this wave equation starting with Maxwell's equations in free space (i.e. no charges and no currents).

Part B

A closed rectangular loop of wire (dimensions b x a) is moving with a uniform velocity v, away from a very long straight wire that carries a current I. Long, current-carrying wire lies in the same plane as that defined by the rectangular wire loop. The side of the closed wire loop with length b is parallel the long current carrying wire. The variable x gives the distance from the current carrying wire to the closest edge of the closed rectangular loop.



(ii) Derive an expression for the magnetic flux passing through the loop of wire as a function of x.

(iii) Hence (or otherwise) determine the induced EMF around the rectangular wire loop as a function of x.