THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

NOVEMBER 2007

PHYS2050

Electromagnetism

Time Allowed – 2 hours

Total number of questions – 5

Part A has 2 questions.

Part B has 3 questions

Answer ALL questions

All questions ARE of equal value

Answer questions from Part A and Part B in separate books

Candidates may not bring their own calculators.

The following materials will be provided by the Enrolment and Assesment Section: Calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

PPS

CARTESIAN: $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}$; $\quad d\mathbf{r} = dx\,dy\,dz$

Gradient $$\nabla T = \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl $$\nabla \times \mathbf{v} = \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Laplacian $$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

SPHERICAL: $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\, d\phi\,\hat{\boldsymbol{\phi}}$; $\quad d\mathbf{r} = dl_r dl_\theta dl_\phi = r^2 \sin\theta\, dr d\theta d\phi$

Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

Laplacian $$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}$$

CYLINDRICAL: $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}$; $\quad d\mathbf{r} = r dr\, d\phi\, dz$

Gradient $$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}$$

Divergence $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl $$\nabla \times \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right)\hat{\mathbf{z}}$$

Laplacian $$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} + \frac{\partial^2 T}{\partial z^2}$$

VECTOR IDENTITIES

TRIPLE PRODUCTS

(1) $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

(2) $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$

PRODUCT RULES

(3) $\nabla(fg) = f\nabla g + g\nabla f$

(4) $\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$

(5) $\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)$

(6) $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$

(7) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(8) $\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$

SECOND DERIVATIVES

(9) $\nabla\cdot(\nabla\times\mathbf{v}) = 0$

(10) $\nabla\times(\nabla T) = 0$

(11) $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$

FUNDAMENTAL THEOREMS

Gradient Theorem $\quad \int_{\mathbf{a}}^{\mathbf{b}} (\nabla T)\cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a})$

Divergence Theorem $\quad \int\limits_{volume} (\nabla\cdot\mathbf{v})\,d\mathbf{r} = \oint\limits_{surface} \mathbf{v}\cdot d\mathbf{a}$

Curl Theorem $\quad \int\limits_{surface} (\nabla\times\mathbf{v})\cdot d\mathbf{a} = \oint\limits_{boundary\ line} \mathbf{v}\cdot d\mathbf{l}$

Coulomb's Law $\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\hat{\mathbf{r}}$

Electric Field $\mathbf{F} = Q\mathbf{E}$ $\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r)}{r^2}\hat{\mathbf{r}}d\mathbf{r}$

Gauss' Law $\int \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{enclosed}}{\varepsilon_0}$ $\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$

Electric Potential $V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}}\mathbf{E}\cdot d\mathbf{l}$ $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$

Poisson's Eqn $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ $V = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r)}{r}d\mathbf{r}$

Laplace's Eqn:- $\nabla^2 V = 0$ Sphere $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

Cylinder $V(r,\phi) = a_0 \ln r + c_0 + \sum_{k=1}^{\infty}\left(c_k r^k + d_k r^{-k}\right)(a_k\cos k\phi + b_k \sin k\phi)$

Boundary Conditions $V_{above} = V_{below}$ $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$

Legendre polynomials $P_0(x) = 1$ $P_1(x) = x$ $P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$ $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

Stored energy $W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i)$ $W = \frac{1}{2}\int\rho(\mathbf{r})V(\mathbf{r})d\mathbf{r} = \frac{\varepsilon_0}{2}\int E^2 d\mathbf{r}$

Bound charge densities $\rho_b = -\nabla\cdot\mathbf{P}$ $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$

Electric Displacement $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$ $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f\,\text{enclosed}}$

Linear Dielectrics $\mathbf{P} = \chi_e\varepsilon_0\mathbf{E}$ $\mathbf{D} = \varepsilon_0(1+\chi_e)\mathbf{E} = \varepsilon\mathbf{E}$

$\varepsilon = \varepsilon_r\varepsilon_0 = (1+\chi_e)\varepsilon_0$

Boundary Conditions $\mathbf{D}^{\parallel}_{above} - \mathbf{D}^{\parallel}_{below} = \mathbf{P}^{\parallel}_{above} - \mathbf{P}^{\parallel}_{below}$ $D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$

Capacitance $C = \frac{Q}{V}$

Biot-Savart	$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}' \times \hat{\mathbf{s}}}{s^2}$	
Ampere's Law	$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{enc}$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$
Vector Potential	$\mathbf{B} = \nabla \times \mathbf{A}$	$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l}'}{s}$
Magnetic Forces	$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$	$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$
Bound current densities	$\mathbf{J}_b = \nabla \times \mathbf{M}$	$\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$
Boundary Conditions	$\mathbf{B}_{above} - \mathbf{B}_{below} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}})$	
	$\mathbf{A}_{above} = \mathbf{A}_{below}$	$\frac{\partial}{\partial n}\mathbf{A}_{above} - \frac{\partial}{\partial n}\mathbf{A}_{below} = -\mu_0 \mathbf{K}$
Linear Media	$\mathbf{M} = \chi_m \mathbf{H}$	$\mathbf{B} = \mu_0(1+\chi_m)\mathbf{H} = \mu\mathbf{H}$
	$\mu = \mu_r \mu_0 = (1+\chi_m)\mu_0$	
Stored Energy	$W = \frac{1}{2\mu}\int B^2 d\mathbf{r}$	

Faraday's Law $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\phi}{dt}$ $\phi = \int_S \mathbf{B}\cdot d\mathbf{a}$

Inductance $M_{21} = \frac{\mu_0}{4\pi}\oint\oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{s}$ $\varepsilon = -L\frac{dI}{dt}$

Maxwell's Equations

$$\boxed{\begin{array}{ll} \nabla\cdot\mathbf{E} = \dfrac{\rho}{\varepsilon_0} & \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t} & \nabla\times\mathbf{B} = \mu_0\left(\mathbf{J} + \varepsilon_0\dfrac{\partial\mathbf{E}}{\partial t}\right) \end{array}}$$

Polarization Current $\mathbf{J}_p = \frac{\partial\mathbf{P}}{\partial t}$

Maxwell's Equations in Matter

$$\boxed{\begin{array}{c} \nabla\cdot\mathbf{D} = \rho_f \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{E} = -\dfrac{\partial\mathbf{B}}{\partial t} \\ \nabla\times\mathbf{H} = \mathbf{J}_f + \dfrac{\partial\mathbf{D}}{\partial t} \end{array}}$$

for linear media $\left.\begin{array}{l}\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \\ \mathbf{M} = \chi_m\mathbf{H}\end{array}\right\}$ so that $\left.\begin{array}{l}\mathbf{D} = \varepsilon\mathbf{E} \\ \mathbf{H} = \frac{1}{\mu}\mathbf{B}\end{array}\right\}$

PART A

This Part has 2 questions

Answer these questions in one book

QUESTION 1 (20 marks)

a) From the integral form of Gauss' theorem

$$\oint_{surface} \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enclosed}}{\varepsilon_0}$$

derive the differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Show all the steps in the derivation, including the relationship between $Q_{enclosed}$ and ρ.

b) Use Gauss' law to find the electric field inside a uniformly charged sphere of radius R and charge density ρ.

c) Find the energy of a uniformly charged spherical shell of total charge q and radius R using the surface charge on the sphere

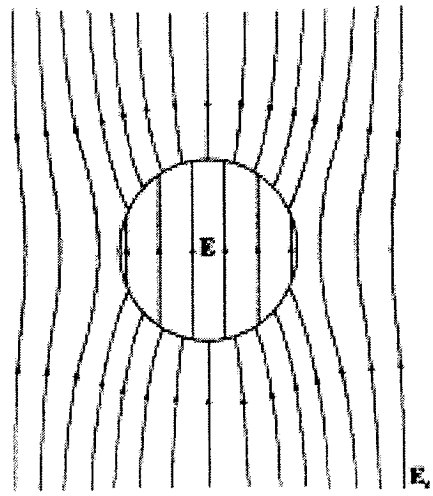
$$W = \frac{1}{2} \int_{surface} \sigma V \, d\mathbf{a}.$$

d) Find the energy of the same uniformly charged spherical shell from the electric field

$$W = \frac{\varepsilon_0}{2} \int_{all\ space} E^2 d\mathbf{r}.$$

QUESTION 2 (20 marks)

A sphere of homogeneous linear dielectric material of dielectric constant ε is placed in an otherwise uniform electric field $\mathbf{E} = E_0\hat{\mathbf{z}}$.



a) Write down the three boundary conditions the potential must satisfy.

b) Using the solution of Laplace's equation for spherical polar coordinates with symmetry in the ϕ direction (from the formula sheet), show that the potential inside the sphere is given by

$$V(r,\theta) = -\frac{3E_0}{\varepsilon + 2} z.$$

You can assume that the coefficients $A_l = B_l = 0$ for $l \neq 1$.

c) Calculate the electric field inside the sphere.

PART B

This Part has 3 questions

Answer these questions in one book

QUESTION 3 (20 marks)

An infinite solenoid of radius R has n turns per unit length, carrying a current I.

a) Use Ampere's law (plus appropriate physical argument) to determine the magnetic field, **B**, both inside and outside the solenoid. What is the direction of **B**?

b) Prove that, in general, the magnetic flux enclosed by any closed loop L, is related to the magnetic vector potential, **A**, by

$$\Phi_B \equiv \int \mathbf{B} \cdot d\mathbf{a} = \oint_L \mathbf{A} \cdot d\ell$$

c) Explain why, in the case of a solenoid, you would expect **A** to have only a ϕ component.

d) Choose a circular loop of radius $r < R$, in a plane at right angles to the axis of the solenoid, and concentric with that axis. Evaluate the magnetic flux through this loop. Hence, using the result from (b), determine **A** as a function of r within the solenoid.

e) Repeat this procedure to determine **A** outside the solenoid.

f) In the region within the solenoid, verify that

$$\nabla \cdot \mathbf{A} = 0 \quad and \quad \nabla \times \mathbf{A} = \mathbf{B}$$

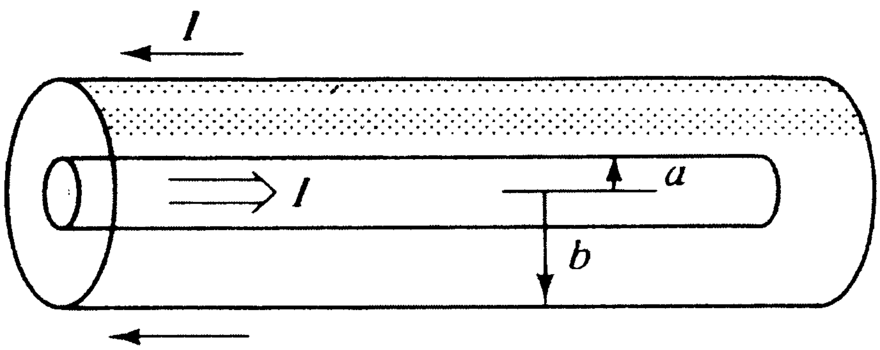
QUESTION 4 (20 marks)

a) An infinitely long cylinder of radius R carries a frozen-in magnetization parallel to the axis, and given by $\mathbf{M} = k\,r^2\,\hat{z}$, where k is a constant, and r is the distance from the axis.

i Find the volume and surface bound currents, $\mathbf{J}_b$ and $\mathbf{K}_b$.

ii Hence find the magnetic field, $\mathbf{B}$, inside and outside the cylinder, as a function of r. (Hint: you will need to take an Amperian loop in the r-z plane.)

b) A long coaxial cable carries a current, I, along its inner surface of radius, a, and back along its outer surface of radius b (see diagram).

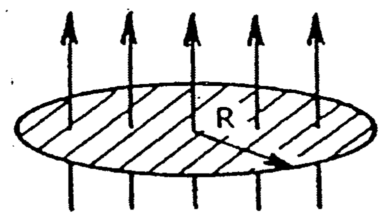


i). Calculate the magnetic field, as a function of r, for $a < r < b$.

ii). Calculate the stored magnetic energy stored in a length l.

iii). Calculate the inductance of the coaxial cable.

QUESTION 5 (20 marks)



a) A uniform electric field, **E**, pointing straight up, fills the shaded region in the diagram. If **E** is increasing at a rate of 10 N C^{-1} s^{-1}, determine, for distances $r < R$

i). the displacement current

ii). the induced magnetic field (magnitude and direction).

b) The classical wave equation for a vector (wave) field, **G**, is

$$\nabla^2 \mathbf{G} = \frac{1}{v^2}\frac{\partial^2 \mathbf{G}}{\partial t^2}$$

where v is the velocity of propagation of the wave. Using Maxwell's equations in free space (i.e. in the absence of free charges and currents), show that the electric field, **E**, satisfies the wave equation, and thereby determine the velocity of propagation of the electric field.