THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

EXAMINATION – JUNE/JULY 2006

PHYS2010 – MECHANICS

Time allowed – 2 hours

Total number of questions – 5

Answer ALL FIVE questions.

The questions are of equal value.

This paper may be retained by the candidate.

The following materials will be provided by the Enrolments and Assessments Section:

Calculators.

All answers must be in ink.

Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

FORMULA SHEET

Damped Harmonic Motion

$$m\ddot{x} + b\dot{x} + kx = 0$$
$$x = Ae^{qt}$$
$$q = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$
$$\gamma = \frac{b}{2m}$$
$$\omega_0^2 = \frac{k}{m}$$

Forced Harmonic Motion

$$m\ddot{x} + b\dot{x} + kx = 0$$
$$x = A\cos(\omega t - \varphi)$$
$$A = \frac{F_0}{m\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + 4\gamma^2\omega^2}}$$
$$\tan\varphi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$
$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$
$$Q = \frac{\sqrt{\omega_0^2 - \gamma^2}}{2\gamma}$$

Central field

$$U_{eff}(r) = \frac{M^2}{2mr^2} + U(r)$$
$$\varphi = \pm\int \frac{\left(M/r^2\right)dr}{\sqrt{2m\left[E - U_{eff}(r)\right]}}$$
$$t = \pm m\int \frac{dr}{\sqrt{2m\left[E - U_{eff}(r)\right]}}$$

Lagrangian

$$L = T - U$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Question 1

Part A

A force is given by:

$$F(x,y,z) = \left(2xy + z^3\right)\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$$

(a) Show that this force is conservative.
(b) Find the potential energy, V.
(c) Determine the work done in moving an object in this force field from (1,-2,1) to (3,1,4).

Part B

Suppose that the force acting on a particle can be expressed in one of the following forms:

(d) $F(\dot{x},t) = f(\dot{x})g(t)$
(e) $F(x,\dot{x}) = f(x)g(\dot{x})$
(f) $F(x,\dot{x},t) = k(x + \dot{x}t)$
(g) $F(x,t) = f(x)g(t)$

In each case, determine whether the equations of motion integrable and if they are, demonstrate how they are integrable and, if possible, integrate them.

Question 2

The equation of motion of a damped harmonic oscillator is given by:

$$m\ddot{x} + c\dot{x} + kx = 0$$

where m is the mass of the particle, c is the damping coefficient and k is the coefficient of the restoring force. The above oscillator is driven by a harmonic driving force given by:

$$F(t) = F_o e^{i\omega t}$$

On solving the equations of motion for this driven damped harmonic oscillator, one finds that the amplitude, A(ω), as a function of frequency is given by:

$$A(\omega) = \frac{F_o/m}{\left[\left(\omega_o^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\right]^{\frac{1}{2}}}$$

(a) Explain the meaning of all the symbols in the equation for A(ω).
(b) Derive an equation for the resonant frequency of the system.
(c) Derive the equation for the amplitude of the driven oscillation at the resonant frequency.
(d) Sketch A(ω) as a function of ω for various values of γ. Be sure to plot and label curves where the behaviour of the system changes.
(e) Discuss how the resonance (and the parameters characterising it) vary with damping.

The phase shift in the driven damped harmonic oscillator is given by:

$$\tan(\phi(\omega)) = \frac{2\gamma\omega}{\omega_o^2 - \omega^2}$$

(f) Explain the meaning of this equation.
(g) Plot ϕ as a function of ω for several values of γ. Label all important parameters on your graph (common points, asymptotes, inflection points etc).
(h) Discuss the features of this graph and how they relate to the resonant frequency.

Question 3

A simple pendulum consists of a mass, m, attached to a rigid massless rod of length, l. The rod plus mass can freely rotate around a pivot point located at the end of the rod opposite the mass. This pendulum hangs in a laboratory under the gravitational field of the earth.

(a) Determine the number of degrees of freedom required to describe the motion of this system.

Select suitable generalised coordinate(s) to describe the state of the simple pendulum.

(b) Derive equations for the kinetic and potential energies of the simple pendulum.
(c) Hence (or otherwise), determine the Lagrangian for this system.
(d) Use Lagrange's equations to determine the equation(s) of motion for this system.
(e) Plot a graph of the potential energy as a function of your generalised coordinate(s).
(f) Plot a velocity phase space portrait describing the motion of this simple pendulum. Label the axes and mark the separatrix.
(g) Describe the behaviour of the system as a function of total energy. Refer to your potential energy plot and the velocity phase space portrait.

A second simple pendulum, identical to the first, is suspended on a pivot attached to the mass of the first pendulum, creating a double pendulum system.

(h) Sketch the double pendulum and use your sketch to define the generalised coordinates describing its motion.
(i) Find the Lagrangian for the double pendulum system.
(j) Derive the equations of motion for the system.

Question 4

A central force is given by:

$$f(r) = -\frac{k}{r^3}$$

where k is a constant and r is the distance from the centre. An object of mass m moves under the influence of this central force.

(a) Show that the potential producing this force is given by:

$$V(r) = -\frac{k}{2r^2}$$

(b) Derive an expression for the effective potential, U(r), of the particle, mass m, moving in this central field.

The form of the equation for the effective potential produces three distinct curves that depend on the values of the parameters: m, the particle mass; h (or l), the angular momentum per unit mass; and k, the force constant.

(c) Sketch the three possible effective potential curves for this central force as a function of radius. Label each curve with the conditions on m, h (or l) and k that are appropriate to the curve.

(d) What type of motion will be experienced by the particle in each of the three cases?

(e) Use Newton's second law (or otherwise) to show that the equation of motion for the particle reduces to:

$$\ddot{r} + \frac{1}{r^3}\left(\frac{k}{m} - h^2\right) = 0$$

(f) Hence show that the orbit equation for the particle is given by:

$$\frac{d^2u}{d\theta^2} - u\left(\frac{k}{mh^2} - 1\right) = 0$$

where u=1/r and θ is the angle in plane polar coordinates.

In a real case, k would be fixed by the nature of the force and m would be fixed by the nature of the particle, thus, the angular momentum per unit mass, h (or l), determines the type of motion observed.

(g) Discuss the physical significance of the transition between the types of motion produced by varying h (or l).

Question 5

Part A

(a) Show that the Lagrangian for a general vibrating system is given by:

$$L = \sum_j \sum_k \frac{1}{2}\left(M_{jk}\dot{q}_j\dot{q}_k - K_{jk}q_jq_k\right)$$

Explain the meaning of all symbols in this equation.

(b) Derive the equations of motion of this system.

(c) Hence show that the n roots of the secular determinant are the squares of the normal frequencies of the system.

Part B

Show your understanding of the above theory by working through an example.

Select an example of coupled harmonic oscillators (your choice!!).

(d) Sketch the system of your choice, clearly labelling the generalised coordinates used to determine the equations of motion of the system.

(e) Find the Lagrangian for your system.

(f) Use Lagrange's equations to derive the equations of motion for your system.

(g) Making any necessary approximations, derive the secular equation for your system.

(h) Solve the secular equation for the normal frequencies of the system.

(i) Sketch the motions that correspond to the normal modes associated with each normal frequency.