THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

EXAMINATION – JUNE/JULY 2005

PHYS2010 – MECHANICS

Time allowed – 2 hours

Total number of questions – 5

Answer ALL FIVE questions.

The questions are of equal value.

This paper may be retained by the candidate.

The following materials will be provided by the Enrolments and Assessments Section:

Calculators.

All answers must be in ink.

Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

FORMULA SHEET

Damped Harmonic Motion

$$m\ddot{x} + b\dot{x} + kx = 0$$
$$x = Ae^{qt}$$
$$q = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$
$$\gamma = \frac{b}{2m}$$
$$\omega_0^2 = \frac{k}{m}$$

Forced Harmonic Motion

$$m\ddot{x} + b\dot{x} + kx = 0$$
$$x = A\cos(\omega t - \varphi)$$
$$A = \frac{F_0}{m\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + 4\gamma^2\omega^2}}$$
$$\tan\varphi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$
$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$
$$Q = \frac{\sqrt{\omega_0^2 - \gamma^2}}{2\gamma}$$

Central field

$$U_{eff}(r) = \frac{M^2}{2mr^2} + U(r)$$
$$\varphi = \pm \int \frac{\left(M/r^2\right)dr}{\sqrt{2m\left[E - U_{eff}(r)\right]}}$$
$$t = \pm m \int \frac{dr}{\sqrt{2m\left[E - U_{eff}(r)\right]}}$$

Lagrangian

$$L = T - U$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$$

Question 1

A wedge of mass M with an angle ϕ is free to slide on a frictionless horizontal table. A solid ball of radius a and mass m is placed on the slope of the wedge. The contact between the ball and the wedge is perfectly rough.

(a) Derive an expression for the total kinetic energy of this system.
(b) Derive an expression for the Lagrangian of this system.
(c) Use Lagrange's equations to find the equations of motion for this system.
(d) Derive an analytic expression for the horizontal acceleration of the wedge with respect to the tabletop.
(e) Discuss the dependence of the acceleration of the wedge on the radius of the sphere.
(f) Find the wedge angle ϕ that produces the maximal wedge acceleration when the mass of the sphere is large compared with the mass of the wedge.

Question 2

Part A

An inverse square law force is given by:

$$f(r) = -\frac{k}{r^2}$$

Substituting this into the orbit equations for a particle acted upon by a central force results in the following equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{mh^2}$$

where u=1/r, m is the mass of the particle and h is the angular momentum per unit mass.

(a) Solve this equation for the orbit of a particle in an inverse square force field.
(b) Show that the solution has the form of a general conic section.
(c) From this (or otherwise) obtain expressions for the following in terms of the parameters describing the particle and its motion:
  (i) the eccentricity of the orbit
  (ii) the perihelion of the orbit
  (iii) the aphelion of the orbit.
(d) Show that for a particle orbiting in a central force, momentum is not conserved.

Part B

Two particles of mass m and M interact via a force that is directed along the line connecting the two particles.

(e) Show that the motion of this physical system can be reduced to the motion of a single (hypothetical) particle in a central force.
(f) Discuss the implications of the non-conservation of momentum in the solution of the central force problem for the two body system.

Question 3

Part A

For the damped harmonic oscillator, the equation of motion can be written as:

$$m\ddot{x} + c\dot{x} + kx = 0$$

(a) explain all terms in this equation.

The equation can be solved by finding the roots of the corresponding subsidiary equation:

$$m\alpha^2 + c\alpha + k = 0$$

This results in three distinct types of solutions that represent three types of physical behaviour: overdamping, critical damping and underdamping.

(b) Explain each of these modes of physical behaviour. Use sketches to illustrate the motion of the particle in each case.

Part B

A new physical system is designed having the following equation of motion:

$$m\ddot{x} + c\dot{x} - kx = 0$$

where m is the mass of a particle and c and k are positive constants.

(c) From the corresponding subsidiary equation (or otherwise), find the solution to this equation of motion (Hint: use the solution of the damped harmonic oscillator as a guide).
(d) How many distinct types of solution and hence physical behaviour, does this system exhibit? (i.e. does it have solutions that correspond to "overdamping, critical damping and underdamping"?).
(e) Discuss the physical reasons for the difference in the physical behaviour of this system compared with the damped harmonic oscillator.

Question 4

A velocity dependent force is given by:

$$\underline{\mathbf{F}} = a\left(\dot{y}\underline{\mathbf{i}} - \dot{x}\underline{\mathbf{j}}\right)$$

where a is a constant and **i** and **j** are the unit vectors in the x and y directions, respectively.

(a) Using Newton's second law of motion (or otherwise), show that the above force results in the following equations of motion for a particle of mass, m, moving in two dimensions(x-y plane):

$$\ddot{x} = -\left(\frac{a}{m}\right)^2 x + C$$
$$\ddot{y} = -\left(\frac{a}{m}\right)^2 y + D$$

where C and D are integration constants.

(b) Show that the general solution to the above equations of motion has the form:

$$x = A\cos\left(\omega_x t + \phi\right) + E$$
$$y = B\sin\left(\omega_y t + \varphi\right) + F$$

Initially, the particle is at the origin (0,0) and it has a velocity:

$$\underline{v_o} = v_o \underline{j}$$

(c) Using these conditions plus restrictions imposed by the equations of motion, show that:
   - (i) The angular frequencies, ω_x and ω_y, are equal.
   - (ii) The phases, ϕ and ψ, are both equal to zero.
   - (iii) The amplitudes are equal and opposite.

(d) Hence, determine the values of all of the unknown constants in the general solution (i.e. find: A, B, ω_x, ω_y, ϕ, ψ, E and F).

(e) Describe the orbit taken by the particle.

(f) Determine the equation of the orbit described by the solution.

Question 5

*Note: In this question, Part A and Part B are **not** connected.*
*Please answer **both** parts.*

Part A

(a) Find the location of the centre of mass of a uniform solid hemisphere of radius a. Show all your working.

Part B

(b) Show that the Lagrangian for a general vibrating system is given by:

$$L = \sum_j \sum_k \frac{1}{2}\left(M_{jk}\dot{q}_j\dot{q}_k - K_{jk}q_jq_k\right)$$

Explain the meaning of all symbols in this equation.

(c) Derive the equations of motion of this system.

(d) Hence show that the n roots of the secular determinant are the squares of the normal frequencies of the system.