1131 T1 2008

Question 1 (marks)

You are cycling, on a long straight path, at a constant speed of 6.0 $m.s^{-1}$. Another cyclist passes you, travelling on the same path in the same direction as you, at a constant speed of $v_b = 9.0$ $m.s^{-1}$. At the instant when she passes you, you realise that she is a friend of yours and you accelerate to catch up to her. You accelerate, starting from t = 0, the time when she passes you, with constant acceleration a = 1.5 $m.s^{-2}$.

a)

i) On a displacement-time graph, sketch your position , $x_a(t)$, and the position of your friend, $x_b(t)$ as functions of time, for time $t < 0$ (i.e. while she is still behind you). Label these sections of the graphs x_a and x_b.

ii) Also sketch displacement-time graphs for you your friend for time $t > 0$. Label these sections of the graphs x_a and x_b as well.

iii) Showing your working, derive both algebraic expressions and quantitative values for the time and distance it takes you to catch up with your friend.

b) In part (a), you accelerated with constant acceleration and overtook her. In this part, you accelerate with forwards acceleration a = 1.5 $m.s^{-2}$ for a time T_1, then decelerate with forwards acceleration a = –1.5 $m.s^{-2}$ for a time T_2. You judge T_1 and T_2 so that when you stop decelerating, you are travelling alongside her at the same speed.

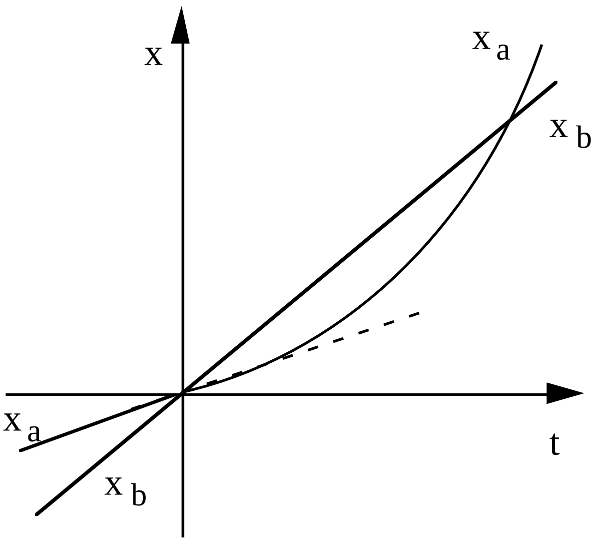
Draw a second displacement-time graph to show this situation. Clearly mark x_a, x_b and the time intervals T_1 and T_2.

c) You are cycling North with velocity $\underline{\mathbf{v}}_1$. Relative to you, the wind appears to be coming from the East. You double your speed to 2 v_1, still in the North direction. The wind has not changed, but now it appears to be coming directly from the North East (ie at 45° ahead of you and from the right).

i) Derive an expresion for the speed of the wind, with respect to the ground, in terms of v_1.

ii) What is the direction of the wind with respect to the ground?

Question 1



iii) $x_b = v_b t$

$x_a = v_0 t + \frac{1}{2} a t^2$

catch up when

$x_a = x_b$

$v_b t = v_0 t + \frac{1}{2} a t^2$

$v_b - v_0 = \frac{1}{2} a t$

$t = \frac{2(v_b - v_0)}{a}$

$= \frac{2(9.0 \text{ m.s}^{-1} - 6.0 \text{ m.s}^{-1})}{1.5 \text{ m.s}^{-2}}$

$= 4.0$ s.

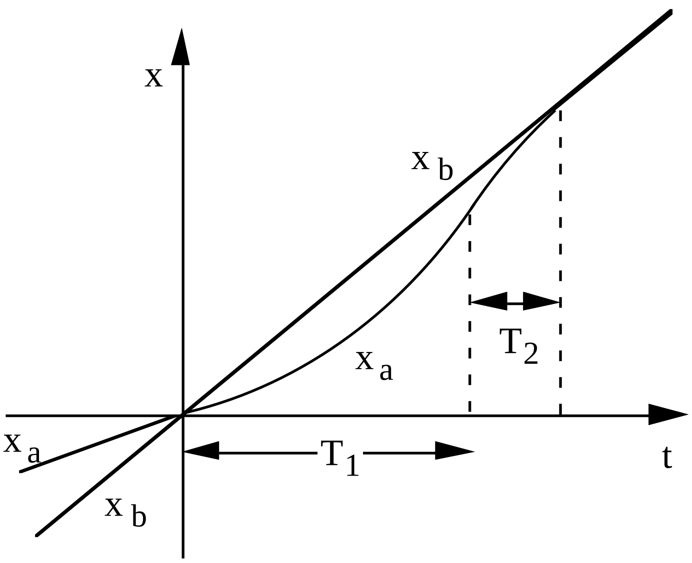
Distance travelled in that time:

$x_b = v_b t$

$= \frac{2v_b(v_b - v_0)}{a}$

$= 36$ m.

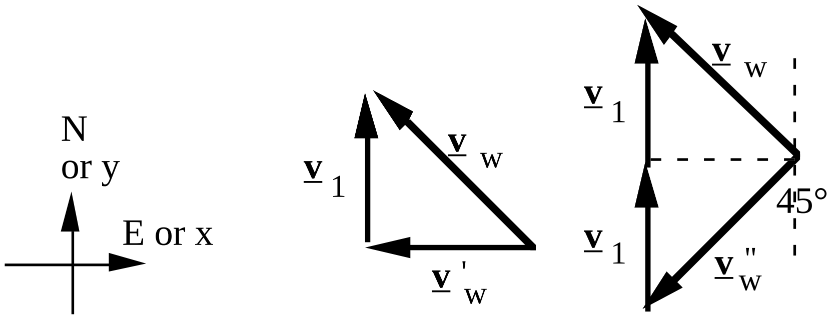
b)



c) Let the apparent velocity of the wind be $\underline{\mathbf{v}}'_W$ in the first case and $\underline{\mathbf{v}}''_W$ in the second case. Its velocity over the ground is $\underline{\mathbf{v}}_W$.

$\underline{\mathbf{v}}_W = \underline{\mathbf{v}}_1 + \mathbf{v}'_W$ and similarly for second case $\underline{\mathbf{v}}_W = 2\underline{\mathbf{v}}_1 + \underline{\mathbf{v}}''_W$

Coordinates first case second case



Here we note that the top triangle in the RH diagram is the same as triangle for case 1. We also see that the large triangle at right is an isosceles triangle. So the answer to (ii) is that the wind comes from the SW.

i) The speed of the wind, from any of these right angle triangles:

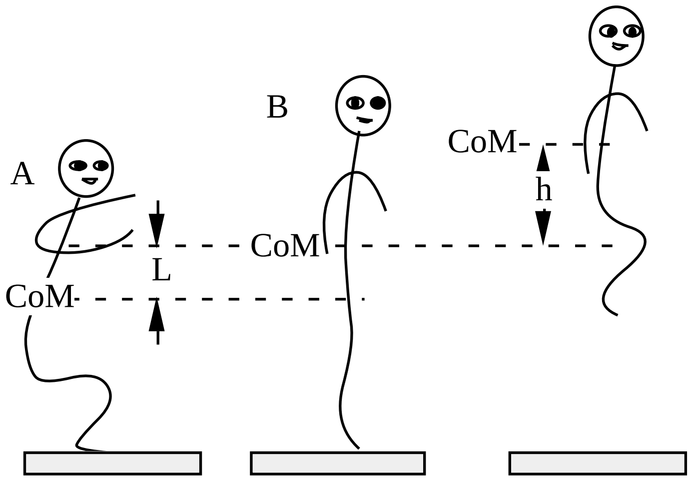
$v_1/v_W = \sin 45°$ so $v_W = \sqrt{2}\, v_1$.

ii) The wind comes from the SE, ie from 45° South of East (or it is going NW, or any equivalent statement. (No verbal justification sought.)

Question 2 (marks)

i) Write Newton's second law in a form that applies to a finite object that is not necessarily rigid, but that has constant mass. If your statement is an equation, state *carefully* the meaning of each term. (For example, do not let the marker wonder "what force?" or "what acceleration?".)

The sketches show successive states of a man jumping vertically in the air. He begins (sketch A) from a stationary position with his legs bent. He then straightens his legs and ankles rapidly: sketch B shows the moment at which his feet leave the ground. The lines marked "CoM" show the height of his centre of mass. Between the first two sketches, his centre of mass rises a distance L. Sketch C shows him at the point where his centre of mass has its maximum height, which is a vertical distance h above its height at the point of take-off. You may neglect air resistance.



ii) Showing your working, and stating any assumptions you make, determine the speed of the man's centre of mass at the moment (B) when his feet leave the ground.

iii) Assume that the vertical acceleration a_{cm} of his centre of mass is constant between A and B. Derive an expression for a_{cm}.

iv) Using your answer to part (i), and thinking carefully, derive an expression for the vertical force **N** exerted by the ground on his feet during the phase A to B.

v) If the man's mass is 70 kg, if L is 0.4 m and h is 0.6 m, what is the downwards force (assumed constant) exerted by his feet during the phase A to B? State any physical law or principle you use in obtaining your answer.

Question 2

i) $\Sigma \underline{\mathbf{F}}_{ext} = m.\underline{\mathbf{a}}_{cm}$

where $\Sigma \underline{\mathbf{F}}_{ext}$ is the total external force, m is the mass of the object and $\underline{\mathbf{a}}_{cm}$ is the acceleration of the centre of mass of the object

OR $\underline{\mathbf{F}}_{ext} = m.\underline{\mathbf{a}}_{cm}$

where $\underline{\mathbf{F}}_{ext}$ is the total external force, m is the mass of the object and $\underline{\mathbf{a}}_{cm}$ is the acceleration of the centre of mass of the object

OR $\Sigma \underline{\mathbf{F}}_{ext} = \frac{d}{dt}(m.\underline{\mathbf{v}}_{cm})$ *OR* $\Sigma \underline{\mathbf{F}}_{ext} = \frac{d}{dt}\underline{\mathbf{p}}_{cm}$ etc

ii) during the jump phase (B $\rightarrow$ C), no external nonconservative forces act, so the mechanical energy associated with the centre of mass is conserved

$U_i + K_i = U_f + K_f$ or $\Delta U + \Delta K = 0$

At C, K = 0, so $\frac{1}{2} m\, v_{cm}^2 = -\Delta U_{grav} = mgh$

so $v_{cm}^2 = 2gh$ so $v_{cm} = \sqrt{2gh}$

iii) motion in one dimension with constant acceleration:

$2a_y L = v_{yf}^2 - v_{yi}^2$. other notations acceptable

$2a_{cm} L = v_{cm}^2$

$a_{cm} = \frac{v_{cm}^2}{2L} = \frac{gh}{L}$

iv) $\Sigma \underline{\mathbf{F}}_{ext} = m.\underline{\mathbf{a}}_{cm}$ applied in the vertical direction gives

$N - mg = m.a_{cm}$

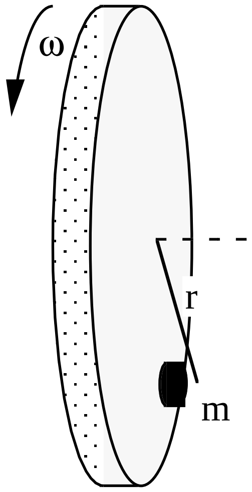
$N = m(g + a_{cm}) = mg\left(1 + \frac{h}{L}\right)$ (direction is upwards)

OR $\underline{\mathbf{N}} = mg\left(1 + \frac{h}{L}\right)$ up.

v) From Newton's third law, this force F has magnitude N but is downwards. $F = mg\left(1 + \frac{h}{L}\right)$ (down). F = 1.7 kN.

Question 3

a)



A small, flat magnet, of mass m, is positioned at a distance r from the centre of a steel disc that rotates with angular velocity ω about a horizontal axis, as shown. The magnitude of the magnetic force between the magnet and the disc is F_m and it is in the normal direction only. The coefficients of static and kinetic friction between the disc and the magnet are respectively μ_s and μ_k respectively, and $\mu_s > \mu_k$. The magnet does not slide when the disc is stationary.

What is the maximum value of ω at which the magnet will not slide on the disk?

(Hint: at which point is it most likely to begin to slide?)

b) With respect to a very large separation, the potential energy of a pair of masses M and m separated by r is U = – GMm/r. The magnitude of the gravitational force between them is $|F| = GMm/r^2$, where r is the distance between their centres and where G is the universal constant of gravitation. Determine the total mechanical energy E of a small satellite (mass m) in a circular orbit of radius r around a planet of mass M in terms of G, M, m and r. (Hint: what is the centripetal force?)

c) Using parts of your answer to (b) or otherwise, derive a relation between the orbital period T and the radius r for a circular orbit of a small mass m about a large mass M. (Hint: what is the circumference?)

d) A satellite, mass m = 120 kg, will be be assembled in the international space station (ISS), which is in a circular orbit about the Earth, at an altitude (i.e. distance above the surface of the Earth) of 350 km. It is then to be moved to a geosynchronous orbit, i.e. one in which it is always directly above a particular point on the equator. How much energy is required to move it from the ISS to the geosynchronous orbit? The radius of the Earth is 6,400 km, its mass is 6.0×10^{24} kg and $G = 6.67 \times 10^{-11}$ N.m^{2}kg^{-2}.

Question 3

While it is not sliding, the magnet undergoes uniform circular motion, so the total force on it is $ma_c = mr\omega^2$ towards the centre. The total force is $\underline{\mathbf{F}}_f + m\mathbf{g}$.

The greatest frictional force is required at the bottom of the circle, where, taking the upwards direction as positive:

$$F_f - mg = \Sigma F = ma_c = mr\omega^2$$

$$\text{so } F_f = mg + mr\omega^2$$

(At the top of the circle, $-F_f - mg = -mr\omega^2$ so $F_f = mr\omega^2 - mg$ and at intermediate angles it has intermediate values.)

From the definition of static friction, the maximum value of $F_{fmax} = \mu_s N$, where N is the normal force, so

$$F_{fmax} = \mu_s N = mg + mr\omega_{max}{}^2$$

Here, the only force in the normal direction is F_m, so $F_m = N$,

$$\mu_s F_m = mg + mr\omega_{max}{}^2$$

$$mr\omega_{max}{}^2 = \mu_s F_m - mg$$

$$\omega_{max} = \sqrt{\frac{\mu_s F_m - mg}{mr}}$$

b) $E = U + K$

$$K = \frac{1}{2}mv^2 \quad \text{centripital force} = m\frac{v^2}{r} = |F_g| = \frac{GMm}{r^2}$$

$$\text{so } mv^2 = \frac{GMm}{r}$$

$$\text{so } E = U + K = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r} = -\frac{GMm}{2r}$$

c) Above we had $mv^2 = \dfrac{GMm}{r}$

The orbit circumference is $2\pi r$, so $v = 2\pi r/T$, so

$$m\left(\frac{2\pi r}{T}\right)^2 = \frac{GMm}{r}$$

$$4\pi^2 r^3 = GMT^2 \qquad \text{(*or equivalent*)}$$

d) For a geosynchronous orbit, the period T (= 23.9 hours) ≅ 24 hours.

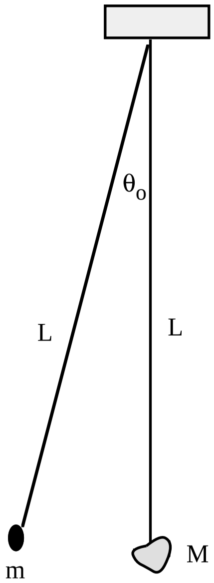
From (c) $$r = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3}$$

Substitution gives $r_{geosynch} = 42{,}000$ km.

From (b) $$\Delta E = -\frac{GMm}{2}\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = \ldots. = 3.0 \text{ GJ}$$

[*This answer will earn full marks. However, it should be noted that, in practice, a much greater quantity of energy will be expended, most of used to move fuel and the container for that fuel.*]

Question 4

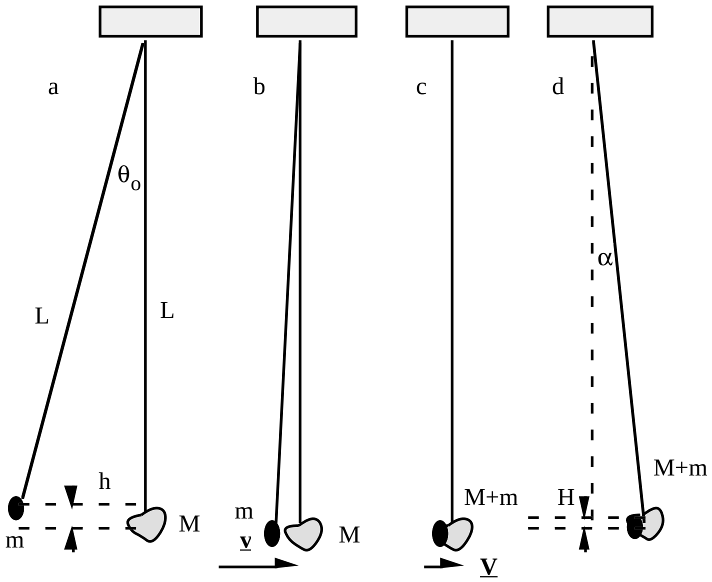


Two light, inextensible strings, each of length L are hung from the same fixed point, as shown. On one, a lump of plasticine (a soft material) of mass M is attached. Initially, it hangs vertically. On the other string, a ball of mass m is attached. Initially, it is stationary, but displaced so that its string makes and angle θ_o with the vertical, as shown. The dimensions of the ball and plasticene are much smaller than L.

The ball is then released. When its string is vertical, it strikes the plasticine and the two remain stuck together.

Showing all working and stating any assumptions you make, derive an expression for the speed of the combined object, immediately after the collision, in terms of the parameters given and g, the gravitational acceleration. Air resistance is negligible.

Question 4



Before the collision (a to b), no nonconservative forces act, so mechanical energy is conserved. Taking the bottom of the path as the zero for U:

$$U_a + K_a = U_b + K_b$$

$$mgh + 0 = 0 + \tfrac{1}{2}mv^2.$$

$$v^2 = 2gh = 2gL(1 - \cos\theta)$$

$$v = \sqrt{2gL(1 - \cos\theta)}.$$

During the collision (b to c), external forces in the horizontal direction are negligible, so momentum is conserved in the horizontal direction. So

$$mv + 0 = (M+m)V$$

$$V = \frac{m}{M+m}v = \frac{m}{M+m}\sqrt{2gL(1 - \cos\theta)}.$$

Question 5

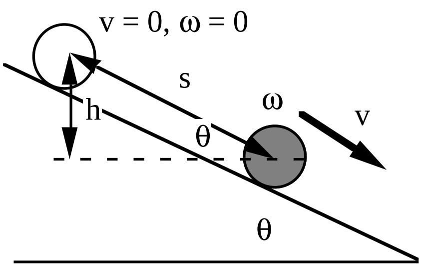
Two cylindrical jars each have radius r. The thickness of the walls is negligible compared with r. When empty, the mass of each jar is m and their radius of gyration k = r (to an approximation sufficient for this problem). One jar is full of water with mass M. The other is full of honey with mass M. They are both placed, stationary, on an inclined plane making an angle θ with the horizontal. Their orientation on the plane allows them to roll to the bottom along the shortest path on the plane. Friction between jars and plane is always sufficient to ensure rolling.

i) The viscosity of the water is sufficiently low that the water does not rotate. Determine the speed of the jar of water after it has rolled a distance s down the plane.

ii) The viscosity of the honey is sufficiently high that the honey rotates as a solid object, at the same rate as the jar. Determine the speed of the jar of honey after it has rolled a distance s down the plane.

The moment of inertia of a hoop is mr^2. That of a disc is $\frac{1}{2}mr^2$, where terms have their usual meaning. Hint: what is the relative velocity at the point of contact during rolling?

Question 5



Because the relative velocity at the point of contact during rolling is zero, nonconservative forces do no work, therefore mechanical energy is conserved.

$$\Delta U + \Delta K = 0$$

$$\Delta K = -\Delta U$$

$$\frac{1}{2}(m+M)v^2 + \frac{1}{2}I\omega^2 = (M+m)gh$$

Rolling $\therefore$ $\omega = v/r$ so

$$(m+M)v^2 + Iv^2/r^2 = 2(M+m)gh$$

i) For water, only the jar rotates, so $I = mk^2 = mr^2$, so

$$(m+M)v^2 + mr^2v^2/r^2 = 2(M+m)gh$$

$$(2m+M)v^2 = 2(M+m)gh$$

$$v = \sqrt{gh\frac{2(M+m)}{2m+M}} \quad \left(= \sqrt{gh\frac{1+m/M}{1/2+m/M}} \right)$$

ii) For honey, both jar and contents rotate, so $I = mr^2 + \frac{1}{2}Mr^2$ so

$$(m+M)v^2 + (m + \frac{1}{2}M)r^2v^2/r^2 = 2(M+m)gh$$

$$(2m+3M/2)v^2 = 2(M+m)gh$$

$$v = \sqrt{gh\frac{2(M+m)}{2m+3M/2}} \quad \left(= \sqrt{gh\frac{1+m/M}{3/4+m/M}} \right)$$