Phys 1131 T1 2004

Question 1. (16 marks)

A scientist is standing at ground level, next to a very deep well (a well is a vertical hole in the ground, with water at the bottom). She drops a stone and measures the time between releasing the stone and hearing the sound it makes when it reaches the bottom.

i) Draw a clear displacement-time graph for the position of the falling stone (you may neglect air resistance). On the diagram, indicate the depth h of the well and the time T_1 taken for the stone to fall to the bottom.

ii) Showing your working, relate the depth h to T_1 and to other relevant constants.

iii) The well is in fact 78 m deep. Take $g = 9.8\ ms^{-2}$ and calculate T_1.

iv) On the same displacement-time graph, show the displacement of the sound wave pulse that travels from the bottom to the top of the well. Your graph need not be to scale.

v) Taking the speed of sound to be $344\ ms^{-1}$, calculate T_2, the time taken for the sound to travel from the bottom of the well to reach the scientist at the top. Show T_2 on your graph.

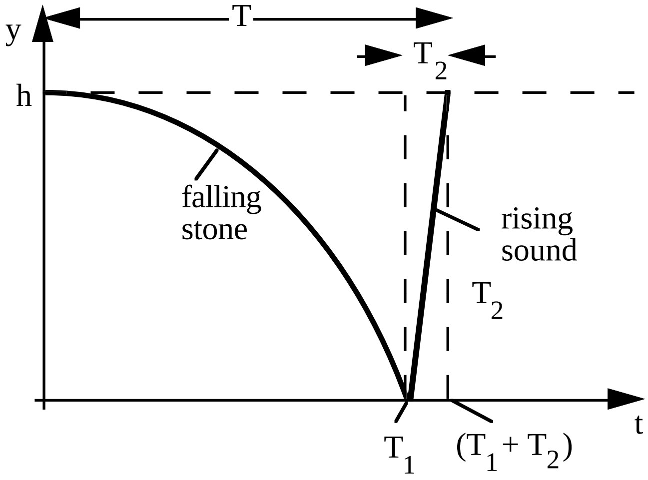
vi) State the time T between release of the stone and arrival of the sound. Think carefully about the number of significant figures.

The scientist, as it happens, doesn't have a stop watch and can only estimate the time to the nearest second. Further, because of this imprecision and because she is solving the problem in her head, she neglects the time taken for the sound signal to reach her. For the same reason, she uses $g \cong 10\ ms^{-2}$.

vii) What value does the scientist get for the depth of the well?

viii) Comment on the relative importance of the errors involved in (a) neglecting the time of travel of sound, (b) approximating the value of g and (c) measurement error.

Question 1.



i) Depending on the choice of origin, the graph might look like these. The algebra below is for the upper case.

ii) $y \;=\; y_o + v_{yo}t + \frac{1}{2}\,a_y t^2$

$\;\;\;= \; h + 0 - \frac{1}{2}\,gt^2$

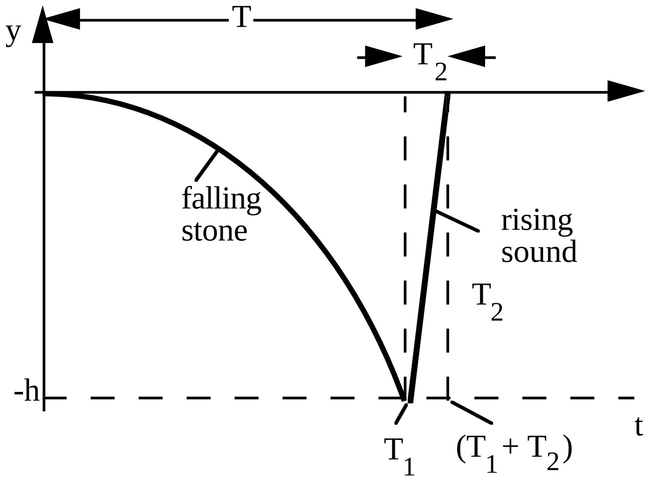
hits the bottom when y = 0, so

$0 \;=\; h - \frac{1}{2}\,gT_1^2$

$\therefore \quad T_1^2 \;=\; \frac{2h}{g}$

$T_1 \;=\; \sqrt{\frac{2h}{g}}$ (4 marks)

iii) $h = 78$ m $\rightarrow$ $T_1 \;=\; 4.0$ s. (1mark)



(5 marks for diagram)

v) speed = distance travelled/time taken, so $T_2 = h/v_s = \; 0.23$ s. (1 mark)

vi) $T = T_1 + T_2 \;=\; 3.99\text{ s} + 0.227\text{ s} \;\rightarrow\; T = \; 4.2$ s. (2 significant figures) (2 marks, incl sig figs)

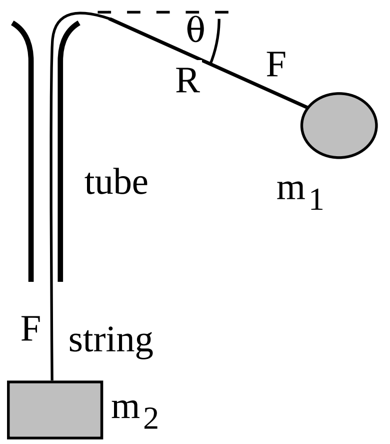
vii) She says $0 \;=\; h - \frac{1}{2}\,gT_1^2$

$\therefore \quad h \;=\; \frac{1}{2}\,gT_1^2 \;\cong\; \frac{1}{2}\,gT^2.$

So she calculates $h \;\cong\; \frac{1}{2}\,(10\text{ms}^{-2})(4\text{ s})^2 \;=\; 80$ m. (3 marks)

viii) Her time estimate is 4.0 ± 0.5 s, an error of 13%. Neglecting the time for the sound to travel (6%) and taking 9.8 ≅ 10 (2%) are small errors by comparison. <She is lucky to have worked out an answer so close to the precise one.> (*Any reasonable comment about the accuracy earns two marks.*) (2 marks)

Question 2. (17 marks)



Two masses, m_1 and m_2, are attached to opposite ends of a string that passes through a tube, whose upper end has been smoothed to reduce the sliding friction with the string. The tube is held vertically and stationary. m_1 is caused to travel in a horizontal circle, in such a way that m_2 does not move. Neglecting the friction between string and tube,

i) Derive an equation for θ in terms of m_1 and m_2.

ii) Derive an expression for the period T of the circular motion of m_1.

iii) State the direction for the normal force N exerted by the tube on the string, and derive an expression for N in terms of F. (You may find a it helpful to draw a diagram)

iv) Now let's consider finite friction between tube and string. Qualitatively, explain how this affects the answer to part (i). Explain your reasoning clearly.

v) Think now about your answer to part (ii). Comment on the limits of the ratio m_1/m_2.

iv) What would be the physical consequence of $m_1 > m_2$ in this experiment, in the case where friction is negligible? A sentence or two is required.

Q2

i) Newton 2 for m_2: $F = m_2 g$ (1 mark)

Newton 2 for m_1 (vertical): $0 = m_2 a_{vert} = F.\sin\theta - m_1 g = m_2 g.\sin\theta - m_1 g$

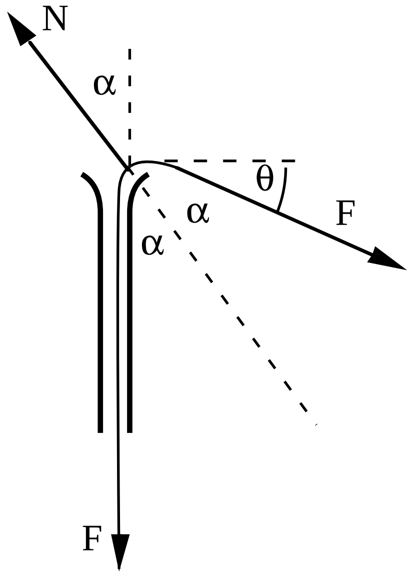
$$\sin\theta = \frac{m_1}{m_2}$$ (3 marks)

ii) Newton 2 for m_1 (horizontal): $m_1 a_{centrip} = m_1 r\omega^2 = F.\cos\theta = m_2 g.\cos\theta$

substitute for r and ω:

$$m_1(R\cos\theta)\left(\frac{2\pi}{T}\right)^2 = m_2 g.\cos\theta$$

$$T = 2\pi\sqrt{\frac{m_1 R}{m_2 g}}$$ (4 marks)



iii) The tension in both ends of the string is F, so, from symmetry, the line of N bisects the angle between the segments of the string. So N is at an angle to the vertical

$\alpha = (90° - \theta)/2$

$N = 2F\cos\alpha$. (3 marks)

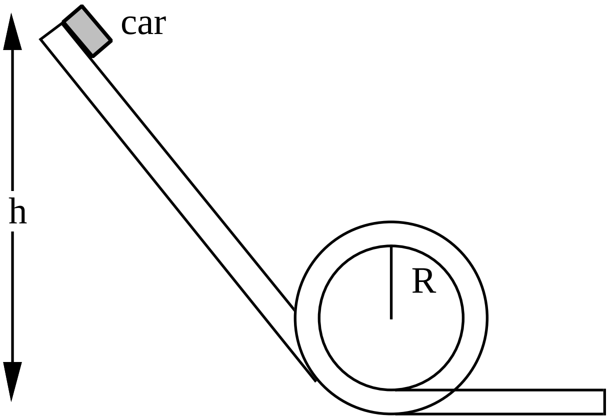
iv) With finite friction, the tension in the string supporting m_1 may be greater than or less than $m_2 g$ by the limit of the static friction. So a range of θ, both greater than and less than the value found above, is possible. (2 marks)

v&vi) $0 < \theta < \pi/2$, so $0 < m_1/m_2 < 1$

If $m_1 > m_2$, then the weight of m_1, which equals the tension in the string, is not great enough both to support m_1 vertically and to provide centripital acceleration. So m_2 rises until it hits the bottom of the tube.

(4 marks for both parts together—their answers may be distributed over both parts.)

Question 3 (13 marks)

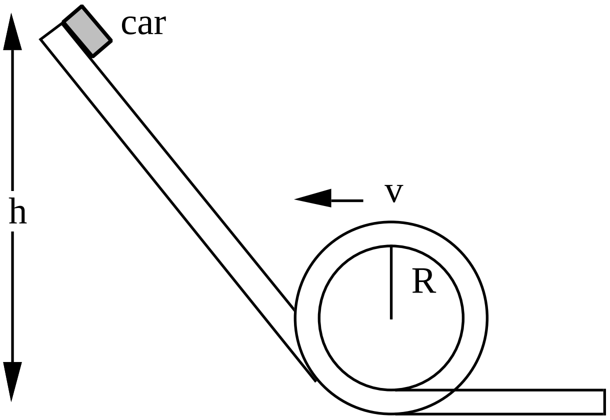


A toy racing car is placed on a track, which has the shape shown in the diagram. It includes a loop, which is approximately circular with radius R. The wheels of the car have negligible mass, and turn without friction on their axle. You may also neglect air resistance. The dimensions of the car are much smaller than R.

i) Showing all working, determine the minimum height h from which the car may be released so that it maintains contact with the track throughout the trip.

ii) If a spherical ball is placed at the height calculated in part (i), will it maintain contact with the track? Explain your answer in a few clear sentences.

Question 3



i) v must be sufficiently great that the centripital force at the top of the loop at least equals the weight of the car. (Faster than this, a downwards normal force is required.)

No non-conservative forces do work, so conservation of mechanical energy applies:

$$U_i + K_i = U_f + K_f$$

$$mgh + 0 = mg.2R + \frac{1}{2} mv^2$$

$$v^2 = 2g(h - 2R)$$ (4 marks)

It loses contact when normal force = 0.

$N + mg = F_{centrip} = mv^2/R$ i.e. falls when $mg = mv^2/R$

$v^2 = gR$ (3 marks)

Therefore it just falls off if h satisfies

$$gR = v^2 = 2g(h - 2R)$$

$5gR = 2\ gh \quad \therefore \quad h_{min} = 5R/2$ (3 marks)

ii) For the marble, some of the initial potential energy is converted into rotational kinetic energy, so there is proportionlly less translational kinetic energy at the top of the loop, so its speed is less. So the centripital force required is less than the weight, so it falls off the track. (3 marks)

Question 4. (17 marks)

i) An apple, attached to a tree a distance of 6370 km from the centre of the Earth, falls to the ground, and appears to accelerate at 9.80 ms^{-2}. The average Earth-moon distance is 3.84×10^8 m. Making the approximation that the Earth is an inertial frame, using these two data and the inverse square law of gravitation, but*without using a value for the gravitational constant G or the mass of the Earth,* determine the period of the moon's orbit around the Earth. Express your answer in days. Give at least one reason why your answer might differ from a lunar month (29.5 days).

ii) The International Space Station has an orbital period of 91.8 minutes. The mass of the Earth is 5.98×10^{24} kg and its radius is 6.37×10^6 m. $G = 6.673\times10^{-11}$ N m^2 kg^{-2}. From these data and the law of universal gravitation, determine the elevation of the station above the Earth and its speed.

i) $$a_g = \frac{F}{m} = \frac{\text{const}}{r^2}$$

When r = 6.37 10^6 m, $a_g \cong g = 9.80\ ms^{-2}$, so const = $a_g r^2 \cong 3.98\ 10^{14}\ m^3s^{-2}$.

$$a_{moon} = r_{moon}\omega^2 = \frac{\text{const}}{r_{moon}^2}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_{moon}^3}{\text{const}}} = \ldots \cong 27.4 \text{ days.}$$ (7 marks)

This is approximately equal to the lunar month.

However, a_g is the acceleration in a frame of reference that accelerates around the Earth's axis of rotation. The real acceleration of the apple is greater than than this by $r_{earth}\omega_{earth}^2$.

Furthermore, during a month, the Earth moves ~360°/13 around the sun, so the lunar month is longer than T by about (14/13).

(2 marks for either. Plus a bonus mark for anyone who gets both.)

ii) $$|F| = \frac{GMm}{r^2} = ma_{centrip} = mr\omega^2$$

$$r^3 = \frac{GM}{\omega^2} = \frac{GMT^2}{2^2\pi^2}$$

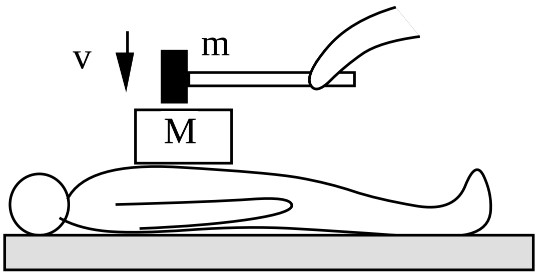
$$r = \sqrt[3]{\frac{GMT^2}{2^2\pi^2}} = 6.74\ 10^6 \text{ m}$$ (6 marks)

$$h = r - r_{Earth} = 370 \text{ km.}$$ (1 mark)

$$v = \frac{2\pi r}{T} = 7.7 \text{ km.s}^{-1}.$$ (1 mark)

Question 5. (12 marks)

a)



In a circus performance, a clown lies on his back with a brick, mass M, on his chest. An assistant uses a hammer with a mass m = 1.0 kg, to crack the brick. The head of the hammer is travelling vertically down at v = 20 ms^{-1}. The mass of the handle is negligible. The collision between hammer and brick is of extremely short duration. However, because the brick cracks at the surface, the collision is completely inelastic.

i) Derive an expression for the velocity V of the brick plus hammer immediately after the collision with the brick.

ii) In an earlier part of the performance, a selection of audience members with different weights has stood on the clown's chest. The deformation of the chest is proportional to the weight of the person standing, and a 100 kg man produces a depression of 30 mm in his chest. Derive an expression for the spring constant of the clown's chest.

iii) The Occupational Health and Safety Officer for the circus decides that the breaking brick trick should not depress the clown's chest more than 30 mm beyond the resting position of the brick before the collision. Derive a value for the required mass M of the brick. You may neglect the gravitational potential energy associated with deformation of the clown's chest.

iv) Express your answer to part (iii) as an inequality. Describe the reason for the direction of the inequality.

Caution. Do not try this exercise at home.

Question 5.

i) During the brief collision, large contact forces act, so external forces are neglected. So momentum of hammer plus brick is conserved. In the vertical direction: $p_{initial} = p_{final}$.

$$mv = (m+M)V,$$

so $$V = \frac{m}{m+M}v. \qquad \text{(3 marks)}$$

ii) From the proportionality of load to deformation, the clown's chest obeys Hooke's law: F = – kx.

Here $k = \dfrac{\text{weight of 100 kg man}}{\text{deformatio}} = \dfrac{980\ \text{N}}{30\ \text{mm}} = 33\ \text{kN.m}^{-1}$. (3 marks)

iii) The clown's chest has been shown to obey Hooke's law, so we assume it acts like a spring, and provides a conservative restoring force. Therefore mechanical energy will be conserved in the deformation that follows the collision. Neglecting gravitational potential energy, we have:

$$U_i + K_i = U_f + K_f$$

$$0 + \tfrac{1}{2}(M+m)V^2 = \tfrac{1}{2}kx_{max}^2 + 0$$

$$M+m\left(\frac{m}{m+M}v\right)^2 = kx_{max}^2$$

$$\frac{m^2}{m+M}v^2 = kx_{max}^2 \qquad (*)$$

$$m + M = \frac{m^2v^2}{kx_{max}^2}$$

$$M = \frac{m^2v^2}{kx_{max}^2} - m = 13\ \text{kg}. \qquad \text{(6 marks)}$$

(Anyone whose sympathy for the clown prompts him/her to point out that this already compresses the chest by 4 mm, and that they OHS Officer should be warned that the total deformation is now 34 mm, should get a bonus mark.)

iv) M > 13 kg. From (*), if M < 13 kg, the energy of the brick after the collision will cause excessive deformation of the clown's chest.

Question 6. (20 marks)

i) A solid sphere, a disc and a hoop are released from rest and roll down an inclined plane, beginning at height h and ending at height 0. Air resistance is negligible. All have the same radius R. Showing all working, *and stating any assumptions you make*, determine the speed v of one of the objects at the bottom of the plane, in terms of its radius of gyration.

ii) The radii of gyration are

$$k_{sphere} = \sqrt{\frac{2}{5}}R \qquad k_{disc} = \sqrt{\frac{1}{2}}R \qquad k_{hoop} = R$$

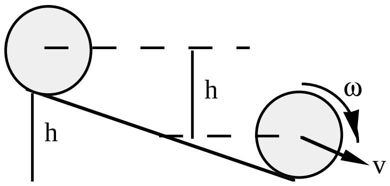
If they are all released at the same time, state the order of their arrival at the bottom, and briefly explain your reasoning.

iii) In two or three clear sentences, explain why one of these objects is faster than another one *in terms of conservation of mechanical energy.*

iv) Using your answer to part (i), state whether a large sphere or a small sphere would roll faster when released from rest on the plane. In one sentence, explain your answer.

v) A can of soup and a can of spaghetti are released form rest and roll down the inclined plane. Which will reach the bottom first? Explain your answer with several sentences of clear reasoning.

vi) The experiment of part (i) is now repeated, using the 'winner' of the race (the fastest of the three objects). This object is released from rest at the top of the plane, at the same time as a toy car, whose (small) wheels turn with no friction on their bearings. Which reaches the bottom first? Explain why in one or two clear sentences.

i) 

Rolling: point of application of friction stationary ∴ non-conservative forces do no work ∴

$$U_f + K_f = U_i + K_i$$

$$0 + \left(\frac{1}{2} M v^2 + \frac{1}{2} I\omega^2\right) = Mgh + 0$$

rolling ∴ $\omega = \frac{v}{R}$ and write $I = Mk^2$

$$\frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \frac{v^2}{R^2} = Mgh$$

$$\frac{1}{2} v^2\left(1 + \frac{k^2}{R^2}\right) = gh$$

$$v = \sqrt{\frac{2gh}{1 + k^2/R^2}}$$ (7 marks)

ii) from (i), v decreases increasing ratio k/R

$$\frac{k_{hoop}}{R} = 1 > \frac{k_{disc}}{R} = \sqrt{\frac{1}{2}} > \frac{k_{sphere}}{R} = \sqrt{\frac{2}{5}}$$

∴ $v_{hoop} < v_{disc} < v_{sphere}$

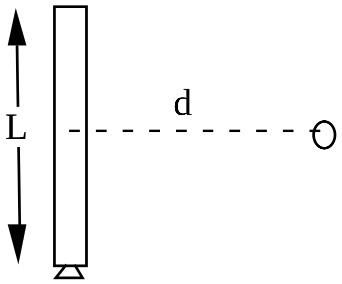
∴ sphere arrives before disc, which arrives before hoop. (3 marks)

iii) During the descent, gravitational potential energy is converted into kinetic energy of two types: translational and rotational. The equation above shows that the ratio of rotational to translational kinetic energy is $(k/R)^2$, so objects with large k/R ratios have proportionally less translational kinetic energy and so lower (translational) speed, all else equal. (2 marks)

iv) the answer to part (i) does not depend on mass m or radius R, so the large and small spheres should travel equally quickly. (1 mark)

v) The wheels of the car are the only part of it that rotates, and they are small, so only a small fraction of the initial gravitational potential energy is converted into rotational kinetic energy, and nearly all is converted into translational kinetic energy. This v will be larger for the car than for an object with a substantial fraction of its kinetic energy in rotation. (2 marks)

vi) The soup wins. The spaghetti behaves nearly as a solid, so it rotates at the same speed as the can. Therefore the spaghetti has rotational kinetic energy. The soup, at least at first, does not rotate with the can. For the spaghetti, the initial gravitational potential energy is converted into translational kinetic energy plus rotational kinetic energy. For the soup, the same quantity of potential energy is converted into translational kinetic energy plus a much smaller quantity of rotational kinetic energy. So the soup has more translational kinetic energy and therefore travels faster. (3 marks)

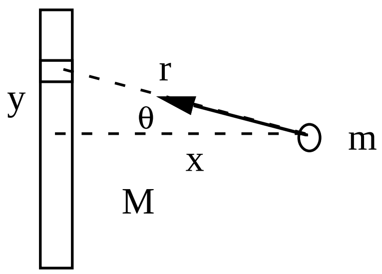


ii) Having run out of fuel, a cosmonaut is initially a distance d = 100 m away from a rocket ship. The rocket ship has mass M, length L = 80 m and is approximately cylindrical. The astronaut is on the plane of symmetry, as shown. They are in deep space, and both are initally stationary with respect to an inertial frame.

Due to gravitational attraction, the cosmonaut (whose mass and dimensions are negligible compared with those of the ship) moves towards the ship.

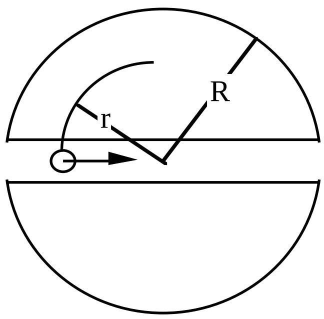
Derive an expression for the time taken for the cosmonaut to reach the ship

iii)



ii) $dF = -G\frac{m.dM}{r^2}\cos\theta = -G\frac{m.dM}{r^2}\cos\theta$

ii) Australia Post is investigating the possibility of delivering mail to distant countries by drilling a hole through the Earth and simply dropping the mail through. Consider a straight hole dug through the Earth and passing through the centre. Showing your working, calculate how long it would take for an object to pass from one end of the hole to the other under the influence of gravity alone. The mass of the Earth is 5.98×10^{24} kg and its radius is 6.37×10^{6} m. $G = 6.673\times10^{-11}$ N m^2 kg^{-2}. For your calculation, assume that the Earth's density is uniform. (Hint: you may use without proof the shell theorem.)



<Shell theorem: the gravitational force due to a thin, uniform spherical shell on another mass is zero if the other mass is inside the shell. If the other mass is outside, it equals the force exerted by a point mass equal to the mass of the shell, but concentrated at its centre.>

When the delivered mail (mass m) is at radius r, the mass of all the shells with smaller radius is

$$M_r = \rho.\tfrac{4}{3}\pi r^3$$

so the force exerted on m at r is

$$F_r = -G\frac{m\rho.\tfrac{4}{3}\pi r^3}{r^2}$$

$$= -Kr \qquad \text{where } K = Gm\rho.\tfrac{4}{3}\pi$$

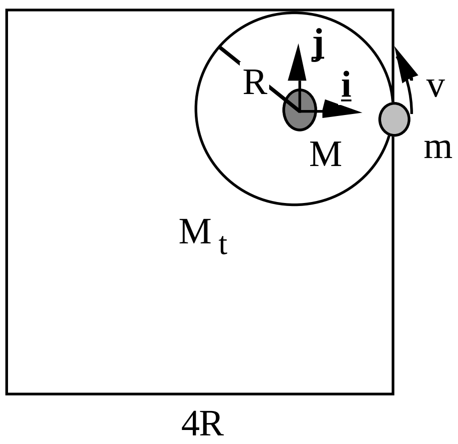
$\therefore$ motion is SHM with $\omega = \sqrt{\dfrac{K}{m}}$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{G\rho.\tfrac{4}{3}\pi}}$$

$$= \frac{2\pi}{\sqrt{GM/R^3}} = \frac{3\pi}{G\rho} = ... = 84 \text{ minutes}$$

$\therefore$ falls through (one half cycle) in 42 minutes *(actually faster for real density profile)*

b)



On a uniform, symmetric square table, of side 4R and mass M_t, stands a woman mass M at the position shown. The woman holds a rope, length R, attached to a dog, mass m, that runs circles of radius R with angular frequency ω around the woman. The dog almost falls off the table, as shown. As part of an even sillier circus act, the woman's partner is going to hold the table above his head, and move so as to balance the table, woman and dog.

i) With respect to unit vectors ($\underline{\mathbf{i}}$,$\underline{\mathbf{j}}$) centred on the woman, derive an expression for the position $\underline{\mathbf{r}}(t)$ of the centre of mass of the dog+woman+table system. The rope has negligible mass, and the dog and woman may be treated as point masses.

ii) Describe in words the path the man must follow in order to stay below the centre of mass. (Hint: it may help to rearrange your expression in (i) so as to separate constant and time-varying terms.)

iii) How long is the man's path? and where is it, quantitatively, in relation to the woman?

Caution. Do not try this exercise at home.

i) With respect to coordinates centred on the woman (where else?), the dog conducts circular motion, so

$\underline{\mathbf{r}}_{dog} = R\cos\omega t\,\underline{\mathbf{i}} + R\sin\omega t\,\underline{\mathbf{j}}$ <*other expressions possible, no mark lost if clockwise*>

$\underline{\mathbf{r}}_{table} = -M_tR\,\underline{\mathbf{i}} - M_tR\,\underline{\mathbf{j}}$

$$\underline{\mathbf{r}}_{cm} = \frac{\Sigma m_i \underline{\mathbf{r}}_i}{\Sigma m_i} = \frac{M_t\underline{\mathbf{r}}_t + M\underline{\mathbf{r}}_{woman} + m\underline{\mathbf{r}}_{dog}}{M_t + M + m}$$

$$= \frac{mR\cos\omega t\,\underline{\mathbf{i}} + mR\sin\omega t\,\underline{\mathbf{j}} - M_tR\,\underline{\mathbf{i}} - M_tR\,\underline{\mathbf{j}}}{M_t + M + m}$$

$$= \frac{R}{M_t + M + m}\Big((m\cos\omega t - M_t)\,\underline{\mathbf{i}} + (m\sin\omega t - M_t)\,\underline{\mathbf{j}}\Big)$$ <*optional step*>

ii, iii) $$\underline{\mathbf{r}}_{cm} = -\frac{M_tR}{M_t + M + m}(\underline{\mathbf{i}} + \underline{\mathbf{j}}) + \frac{mR}{M_t + M + m}(\cos\omega t\,\underline{\mathbf{i}} + \sin\omega t\,\underline{\mathbf{j}})$$

He runs a circular path. Its centre is at $\left(-\frac{M_tR\underline{\mathbf{i}}}{M_t + M + m}, -\frac{M_tR\underline{\mathbf{j}}}{M_t + M + m}\right)$

OR its centre is a distance $\frac{\sqrt{2}M_tR}{M_t + M + m}$ with respect to the woman.

Its radius is $\frac{mR}{M_t + M + m}$ so it has a length $\frac{2\pi mR}{M_t + M + m}$