Question 5 (Marks )

(a) Biot and Savart were able to write down a quantitative law to calculate $B$ in more general cases where the wire is curved, circular, coiled etc. In differential form the law is

$$dB = \frac{\mu_0 I dl \hat{r}}{4\pi r^2}$$

The geometry relevant to this equation is given in the following sketch:

Each length element $dl$ corresponds to a current element $Idl$ which gives a contribution $dB$ to the total magnetic field at a point.
The direction of $dB$ is perpendicular to both $dl$ and $\hat{r}$. It is very handy that the vector cross product gives us precisely what is needed to describe this: Any two vectors, say $A$ and $B$ generate a third vector $C$ pointing perpendicular to the plane containing both $A$ and $B$:

![Diagram showing vectors A, B, and C]

The length of $C$ is given by $|C| = |A||B|\sin \theta$

Therefore $dl\times\hat{r}$ gives the direction of each contribution $dB$ from respective current elements $Idl$

The dependence of $B$ on distance $r$ from the wire was deduced from experiment (by Pierre Simon Laplace 1749-1827) to be

$$dB \alpha \frac{1}{r^2}$$

The full result for $dB$, including constant (to balance units and provide consistency in the electromagnetic system) is

$$dB = \frac{\mu_0}{4\pi} \frac{Id\times\hat{r}}{r^2}$$

If we ignore the direction (vector) information in the above expression we have

$$dB = \frac{\mu_0}{4\pi} \frac{Id}{\sin \theta} \frac{1}{r^2}$$

To get the total $B$ we sum up all the $dB$s by integrating over the spatial variables (we take the current outside the integral as it is assumed constant):

$$B = \frac{\mu_0 I}{4\pi} \int \frac{d\sin \theta}{r^2}$$
We want to find the sum of all contributions $dB_x$ at $P$ due to the current elements $Idl$ at all positions around the ring. This is given by the integral around the entire loop. Each contribution to the total field due to element of current $Idl$ is

$$dB = \mu_0 \frac{Idl \hat{r} \cdot r}{4\pi r^2}$$

where $\hat{r}$ is the unit vector (length one unit) pointing from each $Idl$ to point $P$ and $|r|$ is the distance between each current element and the point $P$.

Notice that $\hat{r}$, and therefore $r$, are always perpendicular to $Idl$. This means that

$$Idl \hat{r} = Idl / \sin \theta = Idl / (\sin \theta = \sin 90^\circ = 1)$$

We also see from the geometry that distance $r^2$ is given by

$$r^2 = a^2 + x^2 \quad \Rightarrow \quad r = \sqrt{a^2 + x^2} \quad \text{(Pythagoras)}$$

Then,
\[ |dB| = \frac{\mu_0 |d\hat{x}|}{4\pi} = \frac{\mu_0 |d\hat{l}|}{4\pi [a^2 + x^2]} \]

We note that the components of dB along the y-direction, dB_y will sum to zero:

\[ dB_y = dB \sin \theta = dB \left( \frac{R}{\sqrt{a^2 + x^2}} \right) \]

\[ = \frac{\mu_0}{4\pi} \left( \frac{|d\hat{l}|}{[a^2 + x^2]} \right) \left( \frac{R}{\sqrt{a^2 + x^2}} \right) \]

\[ r^2 = a^2 + x^2 \quad = \sin \theta \]

and the total field in the x-direction (along axis) is

\[ B_x = \oint dB_x = \oint \frac{\mu_0}{4\pi} \left( \frac{|d\hat{l}|}{[a^2 + x^2]} \right) \left( \frac{a}{\sqrt{a^2 + x^2}} \right) \]

\[ = \oint \frac{\mu_0}{4\pi [a^2 + x^2]^{3/2}} |d\hat{l}| \]

Position x, ring radius a and current I are constants in the problem (along with \( \mu_0 \) and 4\( \pi \)) so that,
\[ B_x = \frac{\mu_0 I a}{4\pi (a^2 + x^2)^{3/2}} \oint dl \]

and

\[ \oint dl = 2\pi a \quad \text{(the circumference of the loop)} \]

so that

\[ B_x = \frac{\mu_0}{4\pi} \frac{I a (2\pi a)}{(a^2 + x^2)^{3/2}} \]

\[ = \frac{\mu_0}{2} \frac{Ia^2}{(a^2 + x^2)^{1/2}} \]

(c) Find the magnetic induction \( B \) due to the whole wire at the point \( X \) located at the centre of the arc, as shown.

\[ \begin{array}{c}
\text{The total B-field at X can be found by adding the contribution from each of the two straight sections and the curved section.}
\\
\text{Each straight section } \Rightarrow \text{ equivalent to half an infinitely long straight wire: } B = \frac{1}{2} \left( \frac{\mu_0 I}{2\pi R} \right)
\\
\text{Curved section } \Rightarrow \text{ equivalent to one quarter of a circular loop: } B = \frac{1}{4} \left( \frac{\mu_0 I}{2R} \right)
\\
\therefore B_{\text{total}} = 2 \cdot \frac{1}{2} \left( \frac{\mu_0 I}{2\pi R} \right) + \frac{1}{4} \left( \frac{\mu_0 I}{2R} \right)
\\
= \left( \frac{1}{2\pi} \right) \frac{\mu_0 I}{R} + \frac{1}{8} \left( \frac{\mu_0 I}{R} \right)
\\
= \left( \frac{1}{2\pi} + \frac{1}{8} \right) \frac{\mu_0 I}{R} = 0.284 \left( \frac{\mu_0 I}{R} \right)
\end{array} \]