SOLUTIONS 1231 T1

Q1. SHM Vibrating Strip

(a)(i) For SHM,

\[ y = A \sin(\omega t + \phi) \]

for amplitude A and angular frequency \( \omega \). Set \( \phi = 0 \).

(ii) The velocity is given by

\[ v = \frac{dy}{dx} = \omega A \cos \omega t \]

The maximum speed \( v_m \) occurs when \( \cos = 1 \),

\[ \therefore v_m = \omega A \text{ with } \omega = 2\pi v \text{, } v = 5 \text{ Hz} \]

\[ |v_m| = 2\pi \cdot 5 \cdot (10^{-3}) \text{m/s} \]

\[ = 0.314 \text{m/s} \]

The acceleration \( a \) in SHM is given by

\[ a = \frac{d^2y}{dt^2} = -\omega^2 A \sin \omega t \]

The maximum value of the acceleration occurs when \( \sin = 1 \) with magnitude

\[ |a_m| = \omega^2 y_m = \omega^2 A \]

\[ \therefore |a_m| = \omega^2 A = (2\pi v)^2 A \]

\[ = (2\pi \cdot 5)^2 \cdot 10^{-2} \text{ms}^{-2} = 9.87 \text{ms}^{-2} \]
(b) Bead mass = 2 g, SHM frequency = 3 Hz.

The acceleration (from part (a)) is

\[ |a_m| = \omega^2 A = (2\pi\nu)^2 A \]

The downward force on the bead due to gravity, \( F_g \), is

\[ F_g = mg = (2 \times 10^{-3}) \times 9.8 = 0.0196 \, N \]

The bead will begin to lose contact when \( |a_m| \geq F_g \), or

\[ (2\pi\nu)^2 A \geq 0.0196 \]

\[ A \geq \frac{0.0196}{(2\pi \cdot 3)^2} \]

\[ A \geq 5.5 \times 10^{-5} \, m = 0.055 \text{mm} \]
2. U-tube oscillations.

The required diagram is:

Where $x$ is the displacement from equilibrium in either arm of the U-tube, and $2h$ is the total column length of liquid.

When liquid is displaced by $x$, l.h.s. moves $O \rightarrow A$, r.h.s. moves $C \rightarrow B$

Excess pressure on whole liquid = excess height $x$ density $x$ g = $2\pi \rho g$

Since pressure = force per unit area,
force on liquid = pressure \times \text{cross-sectional area of tube} \\
= 2x\rho gA \\
(\rho = \text{density of liquid, } A = \text{cross-sectional area of tube})

Excess pressure causes (restoring) force which accelerates liquid ⇒ Newton’s 2nd Law: F = ma.

Total mass of liquid in tube = 2hAρ \\
(2h \text{ is total length of liquid column})

So F = ma becomes:

\[-(2x\rho gA) = (2hAρ)(a)\]

\[\text{force} \quad \text{mass} \times \text{acceleration}\]

(minus sign indicates acceleration directed towards equilibrium position)

Rearranging this we have: 
\[a = -\frac{g}{h}x = -\omega^2 x\]

where \(\omega^2 x\) is the acceleration in SHM with frequency \(\omega\) and

\[\omega = \sqrt{\frac{g}{h}}\]

The period of oscillation is 
\[T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{h}{g}}\]

(b) For the three different liquids in the U-tube the SHM is damped to differing degrees. The motion can be represented graphically by an exponentially decaying sinusoid.

In the diagrams below, the decay envelope has the form \(x \sim e^{-t}\) with \(m = \text{mass and damping constants } a, b\) where \(a > b\) indicates heavier damping in case (i).

[case (iii), undamped SHM is not generally realised in practice but could be observed in special circumstances, e.g. superfluid oscillations in a U-tube or oscillations in a very high Q system – these were mentioned in lectures but students not expected to know it]
(i) Heavy damping (very viscous liquid)

\[ x_m e^{-at/m} \sin \omega t \]

(ii) Light damping (moderate viscosity)

\[ x_m e^{-bt/m} \sin \omega t \]
(iii) Undamped motion (zero viscosity liquid)

\[ x_m \sin \omega t \]
3. Standing waves on string

(i) The two waves given are \( y_1 = 0.20 \sin(2.0x - 4.0t) \) and \( y_2 = 0.20 \sin(2.0x + 4.0t) \) and are of the form:

\[
y = y_m \sin(kx \pm \omega t)
\]

so that \( k = 2.0 \text{m}^{-1} \) and \( \omega = 4.0 \text{s}^{-1} \) by inspection.

Using the identity \( \sin A + \sin B = 2 \sin \left( \frac{A + B}{2} \right) \cos \left( \frac{A - B}{2} \right) \) where \( A \) and \( B \) represent the two wave functions given, the standing wave is \( y_{1+2} \) given by

\[
y_{1+2} = 2y_m \sin kx \cos \omega t = 0.40 \sin(2.0x) \cos(4.0t)
\]

(ii) At position \( x = 0.45 \text{m} \)

\[
y = 0.40 \sin(0.90) \cos(4.0t) = 0.31 \cos(4.0t)
\]

\( \therefore \) maximum amplitude with value \( y = 0.31 \text{m} \) occurs when \( \cos(4.0t) = 1 \)

(iii) For the standing wave pattern

\[
y_{1+2} = 0.40 \sin(2.0x) \cos(4.0t)
\]

we will have nodes at both ends of the string. For such a string fixed at both ends, nodes are also located at positions \( x = n \frac{\lambda}{2} \), so that

\[
\frac{\lambda}{2} = 2 \frac{2\pi}{k} = \frac{\pi}{2.0} m = 1.57m
\]

A standing wave will result when the other end of the string is fixed at \( x \) position

\[
x = n(1.57m) = 1.57m, 3.14m, \ldots (n=1,2,3,\ldots)
\]
(iv) Nodes will occur at \( x = 0, 1.57\text{m}, 3.14\text{m} \ldots \) The maximum amplitude is \( y = 0.40\text{m} \) located at positions mid-way between nodes. Assuming the string is fixed at \( x = 0 \) and \( x = 1.57 \) the maximum amplitude 0.40m is located at \( x = 0.785\text{m} \).

4. Newton’s rings.
The geometry of this arrangement is:

![Diagram of Newton's rings](image)

Ray A: no phase change on reflection from glass block surface into air
Ray B: \( \pi \) phase change on reflection from lens’ lower surface

Transmission of rays: no phase change

(a) The condition for maxima is

\[
2d = (m + \frac{1}{2})\lambda \quad m = 0, 1, 2, 3, 
\]

and for the 5th bright ring (maximum)

\[
2d = (4 + \frac{1}{2})\lambda \quad \text{(note: } m=0 \text{ is the first)}
\]

\[
d = \frac{9}{4}\lambda = \frac{9}{4}(546 \times 10^{-9}) = 1228.5 \times 10^{-9}\text{m}
\]

\[
d = 1228.5\text{nm}
\]

(b) Immersion in the transparent fluid changes the optical path length between lens’ lower surface and glass block

\[
2.d \Rightarrow 2.n.d \quad \text{(optical path length)}
\]

where \( n \) is the refractive index of the fluid.
In air, dark rings occur at

$$2d = m\lambda \quad (m=0,1,2,\ldots)$$

So, in air, the 3rd dark ring is at

$$2d = 3\lambda$$

$$d = \frac{3}{2}\lambda \quad \text{(in air)}$$

If the 5th bright fringe now occupies the position (when in the fluid) that the 3rd dark fringe had (in air), we have

$$(4 + \frac{1}{2})\lambda = 2nd = 2n(\frac{3}{2}\lambda)$$

optical path length

$$4.5 = 3n$$

$$n = 1.5$$

5. Two Slit Interference

(a) Linear separation of fringes on the screen:

Maxima are observed for    $$d \sin \theta = m\lambda.$$    

Where d is slit separation, m is integer. For two adjacent fringes we have

$$d \sin \theta_1 = m\lambda$$

$$d \sin \Theta_2 = (m + 1)\lambda$$

where $\Theta_1, \Theta_2$ are the angular positions of the adjacent fringes. Since the slit-screen separation is 1m we have to a good approximation $\sin \theta \equiv \theta$
\[
\therefore \quad \Delta \theta = \theta_2 - \theta_1 = \frac{\lambda}{d} = \frac{600 \times 10^{-9}}{0.5 \times 10^{-3}} = 1.2 \times 10^{-3} \text{rad}
\]

The linear distance between fringes on the screen will be

\[
\delta = L \Delta \theta = (1m) \times (1.2 \times 10^{-3} \text{rad}) = 1.2 \text{mm}
\]

(b) The intensity pattern on the screen is the product of the interference effect modulated by the diffraction 'envelope'.

**Diffraction minima** occur according to the condition

\[
a \sin \theta = n\lambda \quad \text{(n integer)}
\]

where \(a\) is the slit width. For \(n=1\),

\[
\theta = \frac{\lambda}{d} = \frac{600 \times 10^{-9}}{0.1 \times 10^{-3}} = 6 \times 10^{-3} \text{rad}
\]

The 5th interference fringe in this pattern has zero intensity – it is 'modulated' to zero by the diffraction envelope. The pattern looks like:
(c) The intensity envelope arises because diffraction at each slit modulates the ‘strength’ (intensity) of the interference fringes.

Using a phasor diagram:

The slit pattern has \( N \) sub-rays each differing in phase \( \phi \) by

\[
\phi = \frac{2\pi a}{\lambda} \sin \theta
\]

for slit width \( a \), angular position on screen \( \theta \).

Referring to the diagram,

\[
\phi = \frac{E_m}{R} \quad \text{and} \quad E_0 = 2R \sin \frac{\phi}{2} = \frac{E_m}{\phi/2} \left| \sin \frac{\phi}{2} \right| = E_m \left| \sin \phi/2 \right|^2
\]

and the intensity is

\[
I_0 = I_m \left( \frac{\sin \phi/2}{\phi/2} \right)^2
\]
The intensity of the 3\textsuperscript{rd} fringe relative to the central maximum is

\[ I_3 = I_m \cos^2 \beta \left| \frac{\sin \phi/2}{\phi/2} \right| \]

where \( \beta = \frac{\pi d}{\lambda} \sin \theta = 3\pi \)

\((d \sin \theta = 3\lambda \Rightarrow \sin \theta = 3.6 \times 10^{-3})\)

\(\phi/2 = \frac{\pi d}{\lambda} \sin \theta = \frac{\pi d}{\lambda} (3.6 \times 10^{-3}) = 0.6\pi \)

\[ \therefore I_3 = I_m \cos^2 (3\pi) \left[ \frac{\sin 0.6\pi}{0.6\pi} \right]^2 = 0.255 I_m \]

(d) If there are four (rather than two) slits of equal width, the width of the interference fringes is reduced according to

\[ \Delta \theta = \frac{\lambda}{Nd} \]

where \( \Delta \theta \) is the angular width of interference fringes, \( \lambda \) is the wavelength of illuminating light and \( N \) is the number of slits.

In this case, doubling the number of (equivalent) slits halves the fringe widths to new value 0.3x10\(^{-3}\) rad.