The distance $s$ is given by

$$s = \sqrt{h^2 + (a/2)^2}$$

Coulomb’s law for charges $q_1$ and $q_2$ is

$$F = k_0 \frac{q_1 q_2}{r^2}$$

which gives the magnitude of the force on $+q$ due to $+Q$ and $-Q$

$$|F_{+Q}| = |F_{-Q}| = k_0 \frac{qQ}{h^2} = k_0 \frac{qQ}{\left[ s^2 + (a/2)^2 \right]}$$

and the magnitude of the total force is
\[ |\mathbf{F}_{\text{total}}| = 2k_0 \frac{qQ}{h^2} \sin \theta = 2k_0 \left[ \frac{qQ}{s^2 + (a/2)^2} \right] \sin \theta \]

substituting \( \sin \theta = \frac{a/2}{h} = \frac{a/2}{\sqrt{s^2 + (a/2)^2}} \)

\[ |\mathbf{F}_{\text{total}}| = 2k_0 \frac{qQ}{s^2 + (a/2)^2} \frac{a/2}{\sqrt{s^2 + (a/2)^2}} \]

\[ |\mathbf{F}_{\text{total}}| = k_0 \frac{aqQ}{[s^2 + (a/2)^2]^{3/2}} \]

By symmetry see that \( \mathbf{F}_{\text{total}} \) is parallel to the y-axis, in \( \hat{j} \) direction so that

\[ |\mathbf{F}_{\text{total}}| = k_0 \frac{aqQ}{[s^2 + (a/2)^2]^{3/2}} \hat{j} \]

(b) The magnitude of the force between two charges \( q_1, q_2 \) is given by Coulomb’s law:

\[ F = \frac{1}{4\pi \varepsilon_0} \frac{q_1 q_2}{r^2} \quad (1) \]

where \( \varepsilon_0 \) is the permittivity of free space. The force can be seen to arise from the electric field around charge \( q_i \)

\[ E = \frac{F}{q} = \frac{1}{4\pi \varepsilon_0} \frac{q_i}{r^2} \quad (2) \]
If the charges are immersed in a medium other than free space, $\varepsilon_0$ is replaced in expressions (1) and (2) by the permittivity value $\varepsilon$ of that medium. Thus, permittivity is a scaling factor for electric field strength in a medium and quantifies the electrostatic behaviour of the medium.

(c) Dielectric materials with high permittivity values are used in the production of capacitors. For example, in a parallel plate capacitor, the electric field between the plates carrying surface charge density $\sigma$ is

$$E = \frac{\sigma}{\varepsilon}$$

a dielectric material with permittivity $\varepsilon >> 1$ inserted between the plates lowers the electric field between the plates, thus allowing greater charge to be stored at a given potential difference, i.e. a greater value of $C$ (farads) is obtained for a given geometry.

Other possible engineering examples: dielectric in a coaxial cable; the $\text{SiO}_2$ layer in CMOS microelectronics etc.
QUESTION 2  (Marks 16)

(a) The charge density varies with radius given by

\[
\rho(r) = -\frac{5\rho_0}{2} \left[ 1 - \frac{r^2}{R^2} \right]
\]

where \( \rho_0 = \frac{Q}{(4/3 \pi R^3)} \).

(i) The volume of a spherical shell of charge of thickness \( dr' \) at radius \( r' \) away from the nucleus the is

\[
dV = 4\pi r'^2 dr'
\]

and the charge in this shell is

\[
dq = \rho(r')dV = 4\pi r'^2 \rho(r')dr'
\]

Integrating from \( r' = 0 \) to \( r' = r \),

\[
q(r) = \int dq = \int_0^r 4\pi \rho(r') r'^2 dr' = 4\pi \int_0^r \frac{5\rho_0}{2} \left(1 - \frac{r'^2}{R^2}\right) r'^2 dr'
\]

\[
= -10\pi \rho_0 \left[ \frac{1}{3} r'^3 - \frac{r'^5}{5R^2} \right]_0^r
\]

\[
= -10\pi \left( \frac{Q}{4\pi R^3/3} \right) \left( \frac{1}{3} r^3 - \frac{r^5}{5R^2} \right)
\]

\[
= -\frac{5r^3 Q}{2R^2} + 3\frac{r^5}{2R^5}
\]
The total charge as a fn. of position is charge on nucleus $Q$ plus the electronic contribution:

$$ q(r)_{\text{total}} = Q_{\text{nuc}} + q(r) = Q + \left( -\frac{5r^3 Q}{2R^2} + \frac{3r^5}{2R^5} \right) $$

$$ q(r)_{\text{total}} = Q_{\text{nuc}} + q(r) = Q \left( 1 - \frac{5r^3}{2R^2} + \frac{3r^5}{2R^5} \right) $$

The E-field around the nucleus is symmetric: $\mathbf{E} = E(r)\mathbf{\hat{r}}$. Gauss’ law gives

$$ \int_{\text{surface}} \mathbf{E} \cdot d\mathbf{A} = \int_{\text{surface}} E(r)\mathbf{\hat{r}} \cdot d\mathbf{A} = \int_{\text{surface}} E(r) \, dA = E(r) \int_{\text{surface}} dA = E(r)(4\pi r^2) = \frac{\Sigma q}{\varepsilon_0} $$

(ii) In the region $0 < r < R$,

$$ E(r) = k_0 \frac{q(r)}{r^2} = k_0 \frac{q}{r^2} \left( 1 - \frac{5r^3}{2R^2} + \frac{3r^5}{2R^5} \right) $$

At radius $r = R$ the charge is

$$ q_R = Q \left( 1 - \frac{5r^3}{2R^2} + \frac{3r^5}{2R^5} \right) = 0 $$

as it must be for a neutral atom. By Gauss’ law therefore, we must have $E(r) = 0$ for $r \geq R$. 


Question 3  (Marks 10)

The diagram shows a circular ring of uniform electric charge of radius $a$. The total charge on the ring is $Q$ coulombs. Derive an expression in terms of $z$ and $a$ for the electric potential at a point $P$ vertically above the centre of the ring, $0$, as shown.

![Diagram of a circular ring with a point $P$ above the centre $0$.]

The potential at point $P$ with position vector $\mathbf{r}$ is

$$V = k_0 \int \frac{dq}{r}$$

where $k_0 = \frac{1}{4\pi \varepsilon_0}$. We note from the geometry that

$$r = \sqrt{a^2 + z^2}$$

and then

$$V = k_0 \int \frac{dq}{r} = \frac{1}{4\pi \varepsilon_0} \int \frac{dq}{\sqrt{a^2 + z^2}} = \frac{1}{4\pi \varepsilon_0} \frac{Q}{\sqrt{a^2 + x^2}}$$

$$|\varepsilon_{\text{rms}}| = \frac{7.58 \times 10^{-3}}{\sqrt{2}} V = 5.36 \times 10^{-3} V = 5.36 \text{mV}$$