SOLUTIONS for PHYS3550, General Relativity, Assignment 2 2013 Lecturer: John Webb

Consider the following four different metrics, as given by their line elements:

(1) $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

(2) $ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$ where M is a constant.

(3)

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2\Delta\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dtd\phi \\ &\quad + \frac{(r^2 + a^2) - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \end{aligned}$$

where M and a are constants, $\Delta = r^2 - 2Mr + a^2$, and $\rho^2 = r^2 + a^2\cos^2\theta$

(4)

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2\right]$$

where k is a constant and $R(t)$ is an abitrary function of t.

The metrics above are, in order, the SR, Schwarzschild, Kerr, and Robertson-Walker metrics.

QUESTION (i) For each metric find as many conserved components p_α of a freely falling particle's four momentum as possible.

ANSWER (i) If conserved, clearly it must be that $dp_\alpha/dt = 0$. The general form of the geodesic equation (as given in part (iv)) is

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2}g_{\nu\alpha,\beta}p^\nu p^\alpha$$

Since in general, $p^\nu p^\alpha \neq 0$, the geodesic equation illustrates that if $g_{\nu\alpha,\beta} = 0$, it must be that $dp_\beta/d\tau = 0$.

(1) $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$g_{\nu\alpha} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $g_{\nu\alpha,\beta} = 0$ for all β.
Thus all p_β are conserved in Euclidean space.

(2) $ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, so

$$g_{\nu\alpha} = \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and there are 5 non-zero components of $g_{\nu\alpha,\beta}$ (all the rest are zero):
$g_{tt,r} = -2M/r^2$
$g_{rr,r} = 2M/(r-2M)^2$
$g_{\theta\theta,r} = 2r$
$g_{\phi\phi,r} = r\sin^2\theta$
$g_{\phi\phi,\theta} = 2r\sin\theta\cos\theta$
Thus $p_t,\ p_\phi$ are conserved.

(3)

$$
\begin{aligned}
ds^2 &= -\frac{\Delta - a^2\Delta\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dtd\phi \\
&\quad + \frac{(r^2+a^2) - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2
\end{aligned}
$$

where M and a are constants, $\Delta = r^2 - 2Mr + a^2$, and $\rho^2 = r^2 + a^2\cos^2\theta$. This is the Kerr metric, and we know the symmetry is axial and not spherical. Inspecting the metric components above, we see that the coefficients of dt^2, dr^2, $dtd\phi$, $d\phi^2$, $d\theta^2$ are ALL functions of r and θ, i.e.
$g_{tt,r},\ g_{rr,r},\ g_{t\phi,r},\ g_{\phi\phi,r},\ g_{\theta\theta,r} \neq 0$, and
$g_{tt,\theta},\ g_{rr,\theta},\ g_{t\phi,\theta},\ g_{\phi\phi,\theta},\ g_{\theta\theta,\theta} \neq 0$. All the rest are zero.
Thus p_t, p_ϕ are conserved.

(4) $ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2\right]$
where k is a constant and $R(t)$ is an abitrary function of t.

For this metric, clearly
$g_{rr,t},\ g_{rr,r} \neq 0$
$g_{\theta\theta,t},\ g_{\theta\theta,r} \neq 0$
$g_{\phi\phi,t},\ g_{\phi\phi,r},\ g_{\phi\phi,\theta} \neq 0$
Thus p_ϕ is conserved.

QUESTION (ii) Put (1) in the form $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
From this argue that (2) and (4) are spherically symmetric.
Does this increase, decrease, or leave unchanged, the number of conserved components p_α?

ANSWER (ii) First part standard derivation, from lecture notes.

The spherical symmetry in the Cartesian form is obvious (ds^2 is independent on choice of reference frame orientation). (ii) and (iv) clearly have the same form as $ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, varying only in their coefficients. The spherical symmetry in all 3 metrics is therefore also clear. For metric (i) in polar co-ordinates,

$g_{\theta\theta,r} = 2r$
$g_{\phi\phi,r} = 2r\sin^2\theta$
$g_{\phi\phi,\theta} = 2r^2\sin\theta\cos\theta$
Thus p_t, p_ϕ are conserved. In transforming from Cartesian to polar co-ordinates, the number of conserved components is *decreased* from 4 to 2.

QUESTION (iii) It can be shown that for metrics (2)–(4), a geodesic which begins with $\theta = \pi/2$ and $p^\theta = 0$, i.e. one which begins tangent to the equatorial plane, always has $\theta = \pi/2$ and $p^\theta = 0$.
For cases (2)–(4), use the equation $\overrightarrow{p}.\overrightarrow{p} = -m^2$ to solve for p^r in terms of m, other conserved quantities, and known functions of position.

ANSWER (iii) $\overrightarrow{p}.\overrightarrow{p} = -m^2 = g_{\alpha\beta}p^\alpha p^\beta$.

Metric (2):
The non-zero components are:
$g_{tt} = -(1-2M/r)$
$g_{rr} = (1-2M/r)^{-1}$
$g_{\theta\theta} = r^2$
$g_{\phi\phi} = r^2\sin^2\theta$
Since $\theta = \pi/2$, $\sin^2\theta = 1$. And $p^\theta = 0$. Let $A = (1-2M/r)$ for simplicity. Then

$$-m^2 = -A(p^t)^2 + A^{-1}(p^r)^2 + r^2(p^\phi)^2$$

Substitute and re-arrange to get

$$(p^r)^2 = A\left[A(p^t)^2 - r^2(p^\phi)^2 - m^2\right]$$

Metric (3):
Here, $\Delta = r^2 - 2Mr + a^2$, and we now have $\cos\theta = 0$ so $\rho^2 = r^2$. Also $\sin\theta = 1$. After a little algebra, the non-zero components then become:
$g_{tt} = -(1 - 2M/r)$
$g_{rr} = (1 - 2M/r + a^2/r^2)^{-1}$
$g_{\theta\theta} = r^2$
$g_{\phi\phi} = r^2 + a^2(1 + 2M/r)$
$g_{t\theta} = -4Ma/r$
Now re-arrange the following equation for p^r (the 2 terms in p^θ vanish):

$$-m^2 = g_{tt}(p^t)^2 + g_{rr}(p^r)^2 + g_{\phi\phi}(p^\phi)^2$$

i.e.

$$(p^r)^2 = \left[-m^2 + (1 - 2M/r)(p^t)^2 - [r^2 + a^2(1 + 2M/r)](p^\phi)^2\right](1-2M/r+a^2/r^2)$$

(which seems horrible so maybe I made a mistake somewhere..?)

Metric (4):
Following the same procedure, and letting $B = R^2(t)/(1 - kr^2)$

$$-m^2 = -(p^t)^2 + B(p^r)^2 + Br^2(p^\phi)^2$$

so

$$(p^r)^2 = \frac{1}{B}\left[-m^2 + (p^t)^2 - Br^2(p^\phi)^2\right]$$

QUESTION (iv) For metric (4), spherical symmetry implies that if a geodesic begins with $p^\theta = p^\phi = 0$, these remain zero. Use this together with the general form of the geodesic equation

$$\frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$$

(see eq. 7.29 in Schultz), to show that that if $k = 0$, p_r is a conserved quantity.

ANSWER (iv) We are specifically interested in

$$2m\frac{dp_r}{d\tau} = g_{\nu\alpha,r}p^\nu p^\alpha$$

Dropping terms which are zero we are left only with

$$2m\frac{dp_r}{d\tau} = g_{rr,r}(p^r)^2$$

Now $g_{rr} = R^2(1-kr)^{-1}$
so $g_{rr,r} = R^2k(1-kr)^{-2}$
but since $k=0$, $g_{rr,r}=0$, so p_r is conserved.