UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

PHYS3550 General Relativity

Solutions to Assignment 1 2013

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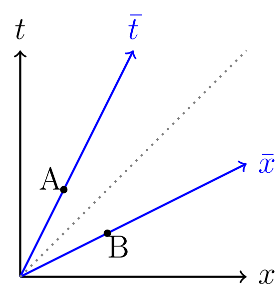
Question 1

Use the Lorentz transformation formulae to derive: (a) The time dilation formula, (b) The Lorentz contraction formulae. Do this by identifying the pairs of events whose separations (in time or space) are to be compared, and then using the Lorentz transformation to accomplish the algebra that was used for the invariant hyperbolae.

Solution



(a) The equation of the $\bar{t}$ axis is $x = vt$, so an event A lying on the $\bar{t}$ axis has coordinates measured in O (i.e. with respect to another event at the origin of O) $(t, x) = (t, vt)$. Using this in the Lorentz transformation formula gives the time dilation formula:

$$\bar{t} = \frac{t - vx}{\sqrt{1 - v^2}} = \frac{t - vvt}{\sqrt{1 - v^2}} = \frac{t(1 - v^2)}{\sqrt{1 - v^2}} = t\sqrt{1 - v^2}$$

(b) The equation of the $\bar{x}$ axis is $t = vx$, so an event B lying on the $\bar{x}$ axis has coordinates $(t, x) = (vx, x)$. Using this in the Lorentz transformation formula gives the Lorentz contraction:

$$\bar{x} = \frac{-vt + x}{\sqrt{1 - v^2}} = \frac{-vvx + x}{\sqrt{1 - v^2}} = \frac{x(1 - v^2)}{\sqrt{1 - v^2}} = x\sqrt{1 - v^2}$$

Question 2

Make a careful sketch or graph as asked below, indicating how you derived or constructed your results for each part.

(a) In the space-time diagram of O, draw the basis vectors $\vec{e}_0$ and $\vec{e}_1$.

(b) Draw the corresponding basis vectors of observer $\overline{O}$, who moves with speed 0.6 in the positive x direction relative to O.

Solution

Graph paper is useful here. (a) is simple of course, $\vec{e}_0$ and $\vec{e}_1$ have unit length. From v=0.6 you can calculate the lengths of $\vec{e}_{\bar{0}}$ and $\vec{e}_{\bar{1}}$ as illustrated in O, and the angles between the axes. Work out angles from γ or use unit vectors and invariant hyperbolae, and plot or sketch on graph paper.

Question 3

The following matrix gives a Lorentz transformation from O to $\overline{O}$:

$$\begin{pmatrix} 1.25 & 0 & 0 & 0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

(a) What is the velocity (speed and direction) of $\overline{O}$ relative to O?
(b) What is the inverse matrix to the given one?
(c) Find the components in O of a vector $A \underset{\overline{O}}{\longrightarrow} (1, 2, 0, 0)$

Solution

See the similar question given as an example on pages 38-40 on Schutz 2nd edition.

Answers are:

(a) $(0.75/1.25) = 0.6$ in the z direction.

(b)

$$(\Lambda^{\bar{o}}_{o})^{-1} = \begin{pmatrix} 1.25 & 0 & 0 & -0.75 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -0.75 & 0 & 0 & 1.25 \end{pmatrix}$$

(c) $A \underset{O}{\longrightarrow} (1.25, 2, 0, -0.75)$

Question 4

(a) Find the energy, rest mass, & three-velocity $\vec{v}$ of a particle whose four-momentum has the components (4,1,1,0) kg.

(b) The collision of two particles of four-momenta

$$\vec{p}_1 \longrightarrow (3, -1, 0, 0)\, kg, \quad \vec{p}_2 \longrightarrow (2, 1, 1, 0)\, kg$$

results in the destruction of the two particles and the production of three new ones, two of which have four-momenta

$$\vec{p}_3 \longrightarrow (1, 1, 0, 0)\, kg, \quad \vec{p}_4 \longrightarrow (1, -1/2, 0, 0)\, kg$$

Find the four-momentum, energy, rest mass, and three-velocity of the third particle produced. Find the CM frame's three-velocity.

Solution

(a) 4-momentum $(E, p^1, p^2, p^3) = (4, 1, 1, 0)$ so $E = 4\, kg$.
Rest mass is obtained from $-m^2 = \vec{p} \cdot \vec{p}$, giving $m = \sqrt{14} = 3.74\, kg$.
$p^0 = E\gamma = 4$, so $\gamma = 4/\sqrt{14}$.
$p^1 = mv_x\gamma = 1$, so $v_x = 1/4$.
$p^2 = mv_y\gamma = 1$, so $v_x = 1/4$.
$p^3 = mv_z\gamma = 0$, so $v_y = 0$.
So 3-velocity is $(1/4, 1/4, 0)$ or $0.25(e_x + e_y)$.

(b)

Four-momentum of third particle:
Conserve momentum. Momentum before = momentum after.

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 + \vec{p}_5$$

where $\vec{p}_5$ is the one we are interested in.

$$(5, 0, 1, 0) = (2 + p_5^0, 0.5 + p_5^1, 1, 0)$$

so

$$\vec{p}_5 \longrightarrow (3, -1/2, 1, 0)\, kg$$

Energy of third particle: Energy is $p_5^0 = 3\, kg$

Rest-mass of third particle:

$$m = \sqrt{-\vec{p}_5 \cdot \vec{p}_5} = \sqrt{31}/2 = 2.78\, kg$$

Three-velocity of third particle produced:
From the above, particle 5's x component of the four-momentum is $p_5^1 = -1/2$, y component,

$p_5^2 = 1$, z component, $p_5^2 = 0$.
Therefore the three-velocity of the third particle is $(-1/6, 1/3, 0)$ or $\frac{-1}{6}e_x + \frac{1}{3}e_y$.

CM frame's three-velocity:
We saw above that

$$\vec{p}_3 + \vec{p}_4 + \vec{p}_5 = (5, 0, 1, 0)$$

The total energy (of all 3 particles produced) is 5, so from the definition of four-momentum, the three-velocity of the CM frame is $(0, 1/5, 0)$ or $0.2e_y$.

Question 5

(a) Let frame $\overline{O}$ move with speed v in the x-direction relative to O. Let a photon have frequency ν in O and move at an angle θ with respect to O's x-axis. Show that the frequencies in O and $\overline{O}$ are related by

$$\frac{\bar{\nu}}{\nu} = \frac{1 - v\cos\theta}{(1 - v^2)^{1/2}}$$

(b) Even when the motion of the photon is perpendicular to the x-axis ($\theta = \pi/2$) there is a frequency shift. This is called the **transverse Doppler shift**, and arises because of the time dilation. At what angle θ does the photon have to move so that there is no Doppler shift between O and $\overline{O}$?

(c) Use the two equations

$$-\vec{p} \cdot \vec{U}_{obs} = \overline{E}$$
$$E = h\nu$$

to calculate the frequency ratio above.

Solution

Take $\gamma = 1/\sqrt{1 - v^2}$.
Take photon path in the xy plane, i.e. it has no momentum in the z-direction.
$\bar{E}$ is the energy of the photon in $\overline{O}$.
p is the photon's 4-momentum.
U_{obs} is $\overline{O}$'s 4-velocity.

(a) Photon's 4-momentum in O is $(E, m\cos\theta, m\sin\theta, 0)$ or $(E, E\cos\theta, E\sin\theta, 0)$.

Transform 4-momentum to $\overline{O}$,

$$p^{\bar{\beta}} = \Lambda^{\bar{\beta}}_{\alpha} p^{\alpha}$$

So we have

$$p^{\bar{\beta}} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0.75 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ E\cos\theta \\ E\sin\theta \\ 0 \end{pmatrix}$$

Since we want the photon energy (or frequency), we only need the first component, i.e.

$$p^{\bar{0}} = \bar{E} = \gamma E - v\gamma E\cos\theta$$

which produces the required answer,

$$\frac{\bar{\nu}}{\nu} = \gamma(1 - v\cos\theta)$$

(b) If there's no Doppler shift, $\bar{\nu}/\nu = 1$, i.e.

$$\gamma(1 - v\cos\theta) = 1$$

which re-arranges to

$$\cos\theta = \frac{1}{v}(1 - \frac{1}{\gamma})$$

(c)

$$-\vec{p}\cdot\vec{U}_{obs} = \overline{E}$$

$$\vec{p} \underset{\overline{O}}{\longrightarrow} (E, m\cos\theta, m\sin\theta, 0)$$

$$U_{obs} \underset{\overline{O}}{\longrightarrow} (\gamma, v\gamma, 0, 0)$$

Taking the scalar product,

$$\begin{aligned}
-\vec{p}\cdot\vec{U}_{obs} &= E\gamma - m\gamma v\cos\theta \\
\bar{E} &= E\gamma - E\gamma v\cos\theta \\
h\bar{\nu} &= h\nu\gamma(1 - v\cos\theta) \\
\frac{\bar{\nu}}{\nu} &= \gamma(1 - v\cos\theta)
\end{aligned}$$