THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF PHYSICS

PHYS3550 General Relativity – Assignment 2, 2013
Lecturer: John Webb
Date set: 22 May 2013
Due: 7 June 2013, by 5pm
Submit paper solutions only please, not by email.
Please submit solutions either to me in lecture period, or place in my mailbox in room OMB G61 (Administrative Office, School of Physics, Old Main Building). Late submissions will have marks deducted. Solutions will be released on 11 June. No marks will be awarded for submissions after that date.

4 metrics:

Consider the following four metrics, as expressed by their line elements:

(1) $$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

(2) $$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where M is a constant

(3) $$ds^2 = -\frac{\Delta - a^2\Delta\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dtd\phi + \frac{(r^2+a^2) - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where M and a are constants and $\Delta = r^2 - 2Mr + a^2$, and $\rho^2 = r^2 + a^2\cos^2\theta$

(4) $$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

where k is a constant and the scale-factor, $R(t)$, is a function of time, t.

Metric (1) is for Special Relativity. Metrics (2), (3) and (4) are, respectively, the Schwarzschild (non-rotating point-mass), Kerr (rotating point-mass), and Robertson-Walker (isotropic, homogeneous) metrics.

Questions:

In answering the following questions, please carefully consider clarity of explanation and presentation. Those will influence your awarded mark.

(i) For each metric find as many conserved components p_α of a freely falling particle's four momentum as possible.

(ii) Put metric (1) in the form

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

From this argue that metrics (2) and (4) are spherically symmetric. Does this change the number of conserved components p_α and if so, how?

(iii) It can be shown that for metrics (2)–(4), a geodesic which begins with $\theta = \pi/2$ and $p^\theta = 0$, i.e. one which begins tangent to the equatorial plane, always has $\theta = \pi/2$ and $p^\theta = 0$. For metrics (2)–(4), use the equation $\vec{p}.\vec{p} = -m^2$ to solve for p^r in terms of m, other conserved quantities and known functions of position.

(iv) For metric (4), spherical symmetry implies that if a geodesic begins with $p^\theta = p^\phi = 0$, these remain zero. Use this together with the geodesic equation

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2}g_{\nu\alpha,\beta}p^\nu p^\alpha$$

to show that if $k = 0$, p_r is a conserved quantity.