THE UNIVERSITY OF NEW SOUTH WALES  
SCHOOL OF PHYSICS  
PHYS 3510  ADVANCED MECHANICS, FIELDS AND CHAOS  
MID-SESSION TEST - 11 SEPTEMBER 2008

Do both questions.  
Both questions of equal marks.

FORMULA SHEET

Euler-Lagrange equations

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \sum_{i=1}^{m} \lambda_i \alpha_{ik} = Q_k
\]

Canonical Transformations

1) \( F_1(q,Q,t) \)
\[
p_i = \frac{\partial F_1}{\partial q_i} \quad P_i = -\frac{\partial F_1}{\partial Q_i} \quad K = H + \frac{\partial F_1}{\partial t}
\]

2) \( F_2(q,P,t) \)
\[
p_i = \frac{\partial F_2}{\partial q_i} \quad Q_i = \frac{\partial F_2}{\partial P_i} \quad K = H + \frac{\partial F_2}{\partial t}
\]

3) \( F_3(p,Q,t) \)
\[
q_i = -\frac{\partial F_3}{\partial p_i} \quad P_i = -\frac{\partial F_3}{\partial Q_i} \quad K = H + \frac{\partial F_3}{\partial t}
\]

4) \( F_4(p,P,t) \)
\[
q_i = -\frac{\partial F_4}{\partial p_i} \quad Q_i = \frac{\partial F_4}{\partial P_i} \quad K = H + \frac{\partial F_4}{\partial t}
\]

Poisson Bracket

\[
[u,v]_{q,p} = \sum_{x} \left( \frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} \right)
\]

Mathematical identities

\[
\sin^2 Q + \cos^2 Q = 1 \quad \tan^2 Q + 1 = \sec^2 Q \quad 1 + \cos 2\phi = 2 \cos^2 \phi \quad 1 - \cos 2\phi = 2 \sin^2 \phi
\]

\[
\frac{d}{dx} \tan x = \sec^2 x \quad \frac{d}{dx} \cot x = \csc^2 x \quad \frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}
\]

\[
\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2} \quad \frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}
\]
**QUESTION 1.** (7.5 marks)

a) Show that the transformation

\[
Q = \tan^{-1} \left( \frac{\alpha q}{p} \right) \quad \text{and} \quad P = \frac{\alpha q^2}{2} \left( 1 + \frac{p^2}{\alpha^2 q^2} \right)
\]

is canonical. Rearrange the transformation equations to obtain

\[
q = \frac{p}{\alpha} \tan Q \quad \text{and} \quad P = \frac{p^2}{2\alpha} \sec^2 Q.
\]

b) Find the generating function for this transformation of type \( F_3(p,Q) \) using both of the relations

\[
q = -\frac{\partial F_3(p,Q)}{\partial p} \quad \text{and} \quad P = -\frac{\partial F_3(p,Q)}{\partial Q}.
\]

c) If the Hamiltonian is \( H(q,p) = p^2 + \alpha^2 q^2 \), find the new Hamiltonian generated by the transformation \( K(Q,P) \).

d) Find the infinitesimal contact transformation generated by the \( F_2(q,P) \) transformation

\[
F_2(q,P) = \sum_i q_i P_i + \varepsilon \sum_i \left( x_i p_{yi} - y_i p_{xi} \right)
\]

Consider only the coordinate components.

e) What is the physical interpretation of this infinitesimal contact transformation.
QUESTION 2. (7.5 Marks)

A heavy particle slides without friction on circular hoop of radius \( a \), under the influence of gravity. Use the method of Lagrange’s undetermined multipliers to obtain the equations of motion for the system in polar coordinates \((r, \theta)\).

The Euler-Lagrange equations are

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \sum_{i=1}^{m} \lambda_i a_{ik} = Q_k \quad k = 1, 2, \ldots, n.
\]

Show that the equations of motion are

\[
\frac{d}{dt} (mr^2) - mr\dot{\theta}^2 + mg \cos \theta = \lambda
\]

\[
\frac{d}{dt} (mr^2 \dot{\theta}) - mgr \sin \theta = 0
\]

Note that the kinetic energy in polar coordinates is

\[
T = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right).
\]

Using the initial condition \( \dot{\theta} = 0 \) at \( \theta = 0 \), show that the Lagrange multiplier is given by

\[
\lambda = mg(3 \cos \theta - 2).
\]

When does the particle separate from the hoop?