THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

MID-TERM EXAMINATION

APRIL 2014

PHYS3021 Statistical and Solid State Physics - PAPER 1

PHYS3020 Statistical Physics

Time Allowed – 50 minutes
Total number of questions - 2
Answer ALL questions

This exam is worth 7.5% of the final grade for PHYS3021 students
This exam is worth 15% of the final grade for PHYS3020 students

Candidates must supply their own university approved calculator.

Answers must be written in ink. Except where they
are expressly required, pencils may only be used
for drawing, sketching or graphical work.

This paper may be retained by the candidate
FORMULA SHEET

Boltzmann Entropy

\[ S = k \ln W \]

Statistics and Distributions

\[ W_B = N! \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!} \quad \frac{N_j}{g_j} = \frac{N}{Z} e^{-\beta g_j/kT} \quad Z = \sum_{j=1}^{n} g_j e^{-\beta g_j/kT} \]

Maxwell-Boltzmann

\[ W_{MB} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!} \quad \frac{N_j}{g_j} = \frac{N}{Z} e^{-\beta g_j/kT} \quad Z = \sum_{j=1}^{n} g_j e^{-\beta g_j/kT} \]

Fermi-Dirac

\[ W_{FD} = \prod_{j=1}^{n} \frac{g_j^{N_j}}{N_j!(g_j-N_j)!} \quad \frac{N_j}{g_j} = \frac{1}{e^{(\epsilon_j-\mu)/kT} + 1} \]

Bose-Einstein

\[ W_{BE} = \prod_{j=1}^{n} \frac{(N_j + g_j - 1)!}{N_j!(g_j - 1)!} \quad \frac{N_j}{g_j} = \frac{1}{e^{(\epsilon_j-\mu)/kT} - 1} \]

Microcanonical

\[ f_{mc}(q,p) = \frac{\delta(H(q,p) - E)}{\int dq dp \delta(H(q,p) - E)} \]

Canonical

\[ f_C(q,p) = \frac{\exp(-\beta H(q,p))}{Z(N,V,T)} \quad Z(N,V,T) = \int dq dp \exp(-\beta H(q,p)) \]

Grand-canonical

\[ f_G(q,p) = \frac{\exp(\beta(\mu N - H))}{\Xi(\mu,V,T)} \quad \Xi(\mu,V,T) = \sum_{N=0}^{\infty} \epsilon^N Z(N,V,T) \]

Thermodynamic Potentials

Internal energy

\[ U \quad dU = TdS - PdV \]

Enthalpy

\[ H = U + PV \quad dH = TdS + VdP \]

Helmholtz function

\[ F = U - TS \quad dF = -SdT - PdV \]

Gibbs function

\[ G = U - TS + PV \quad dG = -SdT + VdP \]

Statistical Mechanics

 Canonical Ensemble

Internal energy

\[ U = kT \left( \frac{\partial \ln Z}{\partial T} \right)_V \]

Pressure

\[ P = kT \left( \frac{\partial \ln Z}{\partial V} \right)_T \]
Mathematical identities

\[ \ln N! = N \ln N - N \]

\[ \int_{0}^{\infty} \frac{x^3}{e^x - 1} \, dx = \frac{\pi^4}{15} \]

\[ \frac{x}{5} = 1 - e^{-x} \quad \Rightarrow \quad x = 4.96 \]

\[ 1 + y + y^2 + \ldots = \frac{1}{1 - y} \]

\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \alpha \]

\[ \int_{0}^{\infty} e^{\frac{\alpha}{2} x^2} \, dx = \frac{\alpha}{2} \sqrt{\pi} \alpha \]

\[ \sinh(x) = \frac{e^x - e^{-x}}{2} \]

\[ \frac{d}{dx} \sinh(x) = \cosh(x) \]

\[ \frac{d}{dx} \cosh(x) = \sinh(x) \]

\[ \text{csch}(x) = \frac{1}{\sinh(x)} \]

\[ \text{sech}(x) = \frac{1}{\cosh(x)} \]

\[ \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \]

\[ \frac{d}{dx} \tanh(x) = \text{sech}^2(x) \]

\[ \coth(x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \]

\[ \frac{d}{dx} \coth(x) = -\text{csch}^2(x) \]

\[ \int_{-\infty}^{\infty} \frac{e^y \, dy}{(e^y + 1)^2} = 1 \]

\[ \int_{-\infty}^{\infty} \frac{y^2 e^y \, dy}{(e^y + 1)^2} = \frac{\pi^2}{3} \]
QUESTION 1  

(15 marks)

The thermodynamic probability $W$ for a system of distinguishable particles, in which each level has an occupancy $N_j$ and a degeneracy $g_j$, is

$$W = N! \prod_{j=0}^{n} \frac{g_j^{N_j}}{N_j!}$$

The two constraints are:

$$\sum_{j=1}^{n} N_j = N$$
$$\sum_{j=1}^{n} N_j E_j = U$$

(i) By minimising $ln W$ subject to these constraints, with an appropriate choice of Lagrange multipliers $\alpha$ and $\beta$, show that

$$N_j = g_j e^{\alpha + \beta E_j}$$

(6 marks)

(ii) By comparing the statistical definition of entropy with the thermodynamic definition, demonstrate that $\beta = -1/(k_B T)$. (4 marks)

(iii) Use the first constraint to determine $e^{\alpha}$. (3 marks)

(iv) The expression above for $N_j$ is called the Boltzmann distribution. In other physical situations we encounter the Maxwell-Boltzmann distribution, which looks formally the same. In what way is a system described by the Maxwell-Boltzmann distribution physically different from a system described by the Boltzmann distribution? (2 marks)
QUESTION 2 (15 marks)

Consider an ideal gas of particles of mass $m$. The Maxwell-Boltzmann distribution is applicable here. Each level has occupancy $N_j$ and degeneracy $g_j$.

(i) The thermodynamic probability $W$ for the Maxwell-Boltzmann distribution takes the form

$$ W = \prod_{j=0}^{n} \frac{g_j^{N_j}}{N_j!} $$

Use this expression and the statistical definition of entropy to determine the entropy $S$ as a function of the particle number $N$, partition function $Z$, and internal energy $U$. (5 marks)

(ii) The partition function $Z$ for an ideal gas is:

$$ Z = V \left( \frac{2\pi mk_b T}{\hbar^2} \right)^{3/2} $$

where $V$ is the volume. Using the known expressions for the pressure $p$ and internal energy $U$ in terms of $Z$, evaluate $p$ and $U$ explicitly. (5 marks)

(iii) Use your results from (i) and (ii) to work out $S$, and show that the entropy per mol, given by $s = S/n_m$ ($n_m$ is the number of moles), satisfies

$$ s = s_0 + R \ln V + c_v \ln T $$

Make sure you identify the constant $s_0$ and the molar heat capacity $c_v$. (5 marks)