Question 1. Heisenberg’s uncertainty principle (Marks 40).
   a. Formulate briefly the Heisenberg uncertainty principle. Outline, also very briefly its
      relation with the de Broglie relations.
   b. Assume that a particle of mass $m$ propagates along the $x$-axis in the potential

   $\frac{U(x)}{2m} = \frac{k}{\sqrt{2n}} x^{2n}$

   (1)

   where $k$ is a positive constant and $n \geq 1$ is an integer. Using Heisenberg’s uncertainty
   principle, estimate the ground state energy $E$, as well as the averaged kinetic and potential
   energies in this state.

   Hint. To simplify algebraic calculations one can choose units $\hbar = m = 1$. Then the only
   dimensional parameter left is $k$. If you still struggle with the algebra, keep in mind that up to
   a numerical factor the dependence of the energy on $k$ (and $m$) can be recovered from simple
   dimensional analyses.

Question 2. Quantum oscillator (Marks 60).
   Consider the conventional quantum oscillator

   $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m \omega^2 \hat{x}^2}{2}$

   (2)

   a. Write down the expression for its energy spectrum $E_n$.

   Introduce the creation and annihilation operators

   $\hat{a}^+ = \frac{1}{\sqrt{2}} \left( \frac{x}{b} + \frac{b}{dx} \frac{a}{dx} \right)$

   (3)

   $\hat{a}^+ = \frac{1}{\sqrt{2}} \left( \frac{x}{b} - \frac{b}{dx} \frac{a}{dx} \right)$

   where $b = \sqrt{\hbar/m \omega}$ is the magnetic length (for the following calculations it can be
   convenient to choose units in which $b = 1$).

   b. Prove that the Hamiltonian (2) can be rewritten as follows

   $\hat{H} = \hbar \omega (\hat{a}^+ \hat{a} + \hat{\psi})$

   (4)

   Hint: remember the commutation relation

   $[\hat{a}, \hat{a}^+] = 1$

   (5)

   c. Find an explicit expression for the ground state wave function $\psi_0(x)$ of the Harmonic
      oscillator.

   Hint: Remember that Eq.(4) allows one to write the linear first order differential
   equation on $\psi_0(x)$, which solution is straightforward.

   d. Consider the Hamiltonian

   $\hat{H}_1 = \hbar \omega (\hat{a}^+ \hat{a} + \lambda \hat{a} + \lambda^* \hat{a}^+)$,

   (6)

   where $\lambda$ is a complex-valued constant. Find the energy spectrum of $\hat{H}_1$.  