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1(a). The Compton effect involves the scattering of photons by electrons in a target. The experiment involves firing a beam of x-rays of wavelength λ_0 at a target (typically graphite, but often other materials), and looking at the scattered x-rays at some angle θ to the incident path. The wavelength of the scattered x-rays is measured by x-ray diffraction from a crystal with a known atomic spacing. The scattered x-rays show peaks at two wavelengths λ_0 and λ' where $\lambda' = \lambda_0 + h/mc\,(1-\cos\theta)$.

The implication of the Compton experiment is that it demonstrates the particle-like nature of photons, which undergo a two-body collision with the electron just like any other particle would. (n.b. if you measure the wavelength by diffraction, then you see both the wave and particle natures of the photon in one experiment!).

1(b). The incident photon has $E = 200\,\text{keV} = \dfrac{hc}{\lambda_0}$ $\Rightarrow$ $\lambda_0 = \dfrac{6.626\times10^{-34}\times3\times10^{8}}{200\times10^{3}\times1.6\times10^{-19}}$

$= 6.2\,\text{pm}.$

The photon loses 10% of its energy (ie. 20keV), so we can find

$$\lambda' = \frac{6.626\times10^{-34}\times3\times10^{8}}{180\times10^{3}\times1.6\times10^{-19}}$$

$$= 6.9\,\text{pm}.$$

Now $\Delta\lambda = \lambda' - \lambda_0 = 0.7\,\text{pm} = \dfrac{h}{mc}(1-\cos\theta)$

$$\therefore\ 1-\cos\theta = \frac{mc\Delta\lambda}{h} \Rightarrow \cos\theta = 1 - \frac{mc\Delta\lambda}{h}$$

$$= 1 - \frac{9.1\times10^{-31}\times3\times10^{8}\times0.7\times10^{-12}}{6.626\times10^{-34}}$$

$$= 0.71$$

$$\theta = 44.6^\circ$$

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1(c). For a plane wave, there is only one k (and thus only one p) so $\Delta p \sim 0$ (ie. the momentum is very well defined), but the wave has the same amplitude for all x (ie $-\infty \leq x \leq \infty$) and so its completely delocalised (ie $\Delta x \sim \infty$). We can localise the packet by starting to add together a number of waves with differing k. This leads to some distribution in p (in other words some uncertainty Δp), and some localisation of the wave Δx. In order to make Δx smaller, you need to add more and more waves with different k, so that if you have small Δx, you will necessarily have a large uncertainty in momentum Δp.

1(d). $\Delta x \sim 0.1\text{nm}$. $\Delta p \geq \hbar / 2\Delta x$

$$\geq \frac{1.054 \times 10^{-34}}{2 \times 0.1 \times 10^{-9}}$$

$$\geq 5.27 \times 10^{-25} \text{ Kgm/s.}$$

Assuming $p \sim \Delta p$ then $E = \frac{p^2}{2m} = \frac{(5.27 \times 10^{-25})^2}{2 \times 9.1 \times 10^{-31}}$

$$= 1.52 \times 10^{-19} \text{ J} \sim 0.95 \text{ eV.}$$

The typical binding energy of electrons in atoms is $\sim 1-10$ eV
$\therefore$ electrons can be confined in atoms within the limits of the uncertainty principle.

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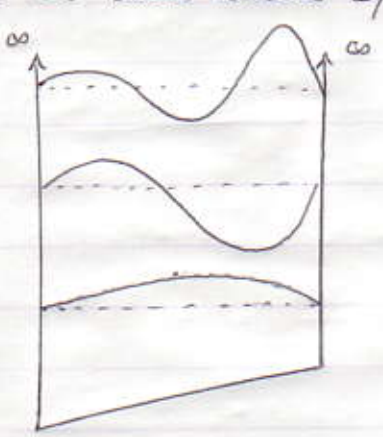
2(a). Inside the well, the Schrödinger equation will be given by

$$\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x) \quad \text{where} \quad k = \frac{\sqrt{2m(\epsilon - V(x))}}{\hbar}$$

Let's now think about k. Towards the right, where $V(x)$ is high, k is low, so is p $(=\hbar k)$, and this means λ is high as $\lambda = h/p$. Conversely, toward the left, where $V(x)$ is low, k is high, so is p and this means λ is low. Also, on the right where p is low, the velocity is low, and so the time spent there is relatively high and the wavefunction amplitude there should be high. By the converse, the wavefunction amplitude on the left of the well should be relatively lower. Finally, the boundary conditions require that $\psi(x) = 0$ at both walls of the well. Thus



2(b)(i)

$$\int_{-\infty}^{\infty} \psi^*\psi \, dx = \int_{-L}^{L} A(L^2-x^2)e^{ikx} A(L^2-x^2)e^{-ikx} \, dx.$$

$$= \int_{-L}^{L} A^2 (L^2-x^2)^2 \, dx.$$

$$= \int_{-L}^{L} A^2 (L^2-x^2)(L^2-x^2) \, dx.$$

$$= A^2 \int_{-L}^{L} L^4 - 2x^2L^2 + x^4 \, dx.$$

$$= A^2 \left[L^4 x - \frac{2}{3}x^3L^2 + \frac{1}{5}x^5 \right]_{-L}^{L}$$

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2(b)(i) $\int_{-\infty}^{\infty} \psi^* \psi \, dx = A^2 \left(L^5 - \frac{2}{3} L^5 + \frac{1}{5} L^5 + L^5 - \frac{2}{3} L^5 + \frac{1}{5} L^5 \right)$

$$= A^2 \frac{16}{15} L^5 = 1$$

$$\therefore A^2 = \sqrt{\frac{15}{16L^5}}$$

2(b)(ii) $\langle p_x \rangle = \int_{-\infty}^{\infty} \psi^* p_x \psi \, dx.$

$$= \int_{-L}^{L} A(L^2-x^2) e^{-ikx} \left(-i\hbar \frac{d}{dx} \right) A(L^2-x^2) e^{ikx} \, dx$$

$$= -i\hbar A^2 \int_{-L}^{L} (L^2-x^2) e^{-ikx} \left[ik(L^2-x^2) - 2x \right] e^{ikx} \, dx$$

$$= -i\hbar A^2 \int_{-L}^{L} (L^2-x^2) \left[ik(L^2-x^2) - 2x \right] dx.$$

if we now split the integral (for convenience)

$$= \hbar k A^2 \int_{-L}^{L} (L^2-x^2)^2 dx + i\hbar 2A^2 \int_{-L}^{L} x(L^2-x^2) \, dx.$$

$$\int_{-L}^{L} (L^2-x^2)(L^2-x^2) \, dx = \int_{-L}^{L} L^4 - 2x^2L^2 + x^4 \, dx.$$

but we already know this from part (i).

$$= \left[L^4 x - \frac{2}{3} x^3 L^2 + \frac{1}{5} x^5 \right]_{-L}^{L} = \frac{16}{15} L^5$$

and $\int_{-L}^{L} x(L^2-x^2) \, dx = \int_{-L}^{L} xL^2 - x^3 \, dx = \left[\frac{1}{2} x^2 L^2 - \frac{1}{4} x^4 \right]_{-L}^{L}$

$$= \frac{1}{2} L^4 - \frac{1}{4} L^4 - \frac{1}{2} L^4 + \frac{1}{4} L^4 = 0.$$

but we knew this anyway as

Is a symetric integral of an odd func.

So $\langle p_x \rangle = \hbar k A^2 \frac{16}{15} L^5 + i\hbar 2A^2 \times 0$

$$= \hbar k. \quad \text{since } A^2 = 15/16L^5$$

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3(a) For the quantum oscillator $V(x) = \frac{1}{2}Kx^2$ but $\omega = \sqrt{\frac{K}{m}} \Rightarrow K = m\omega^2$

$$= \tfrac{1}{2}m\omega^2 x^2$$

So $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = \epsilon\psi(x).$$

and $$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi(x) = \epsilon\psi(x).$$

(b). This is quite easy, the potential is symmetric, therefore the ground state eigenfunction is also symmetric.

(c) $\psi(x) = C_0 e^{-\alpha x^2}$

(i) so $$\frac{d\psi(x)}{dx} = -2\alpha x C_0 e^{-\alpha x^2}$$

$$\frac{d^2\psi(x)}{dx^2} = -2\alpha C_0\left(1-2\alpha x^2\right)e^{-\alpha x^2}$$

if we now put this into the Schrodinger equation, hopefully we will pop out α, so:

$$-\frac{\hbar^2}{2m}\left[-2\alpha\left(1-2\alpha x^2\right)C_0 e^{-\alpha x^2}\right] + \frac{1}{2}m\omega^2 x^2 C_0 e^{-\alpha x^2} = \epsilon C_0 e^{-\alpha x^2}$$

Cancel the $C_0 e^{-\alpha x^2}$ terms $\Rightarrow$ $$\frac{\hbar^2\alpha}{m}\left(1-2\alpha x^2\right) + \frac{1}{2}m\omega^2 x^2 = \epsilon$$

$$\frac{\hbar^2\alpha}{m} - \frac{2\alpha^2\hbar^2 x^2}{m} + \frac{1}{2}m\omega^2 x^2 = \epsilon.$$

now ϵ can't be x-dependent, so we can say $-\frac{2\alpha^2\hbar^2}{m} + \frac{1}{2}m\omega^2 = 0$

or $$\frac{2\hbar^2\alpha^2}{m} = \frac{1}{2}m\omega^2$$

$$\alpha^2 = \frac{1}{2}\times\frac{1}{2}\frac{m^2\omega^2}{\hbar^2}$$

$$\alpha = \frac{1}{2}\frac{m\omega}{\hbar}. \qquad \therefore \epsilon = \frac{\hbar^2\alpha}{m} = \frac{\hbar^2}{m}\frac{m\omega}{2\hbar} = \frac{\hbar\omega}{2}.$$

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3(c)(ii) This is easy $\epsilon = \frac{\hbar^2 \alpha}{m} = \frac{\hbar^2}{m}\frac{m\omega}{2\hbar} = \frac{\hbar\omega}{2}.$

4.(a). 3 regions : M_1, I and M_2

We start by working out the three potentials

$M_1 : V(x) = -V_0/2 \qquad I : V(x) = U_0 \qquad M_2 : V(x) = V_0/2.$

and since $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = \epsilon\,\psi(x)$

$$M_1 \qquad -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \frac{V_0}{2}\psi(x) = \epsilon\,\psi(x)$$

$$I \qquad -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + U_0\psi(x) = \epsilon\,\psi(x)$$

$$M_2 \qquad -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \frac{V_0}{2}\psi(x) = \epsilon\,\psi(x)$$

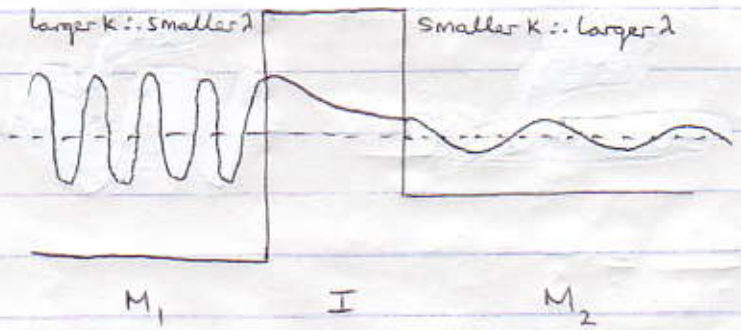
(b). In region M_1, $\epsilon > V(x)$ so the solution should be plane wave, likewise in M_2 except that there's no source on this side and in the I region, $\epsilon < V(x)$ so the solution should be exponential.

Thus $\psi_{M_1}(x) = Ae^{ik_1x} + Be^{-ik_1x} \qquad k_1 = \frac{\sqrt{2m(\epsilon + V_0/2)}}{\hbar}$

$$\psi_I(x) = Ce^{-k_2x} + De^{k_2x} \qquad k_2 = \frac{\sqrt{2m(U_0 - \epsilon)}}{\hbar}$$

$$\psi_{M2}(x) = Fe^{ik_3x} \qquad k_3 = \frac{\sqrt{2m(\epsilon - V_0/2)}}{\hbar}$$

(c).



Note that $k_3 \ll k_1$ because $\epsilon > V(x)$ by more in M_1 than M_2, therefore $\lambda_3 \gg \lambda_1$

Ampl in M_2 < Ampl in M_1 as e^- has to tunnel through the barrier to get from M_1 to M_2.

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5(a) Bohr's 3 postulates are : 1. The electron can only have certain orbits given by $L = mvr = n\hbar$ $(n = 1, 2, 3, \ldots)$
(ie. integer electron wavelengths around an allowed orbit)

2. An electron in an allowed orbit does not emit electromagnetic radiation

3. If an electron changes from one orbit E_i to another orbit E_f, the frequency of the emitted radiation is given by $h\nu = E_i - E_f$ $E_i > E_f$.

In the Rutherford model, the key problem is that the negative electron should be electrostatically attracted to the positive nucleus such that the electron should spiral into the nucleus, bleeding energy as E.M. radiation on the way.

5(b). The centripetal force on an electron orbitting the nucleus will be due to the coulomb force between electron and nucleus, therefore :

$$\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2} \quad (1)$$

now Bohr's 1st postulate tells us that the angular momentum must be quantized : $L = mvr = n\hbar$

$$\Rightarrow v = \frac{n\hbar}{mr} \quad (2)$$

substituting (2) into (1).

$$\frac{m}{r}\frac{n^2\hbar^2}{m^2r^2} = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

$$\frac{n^2\hbar^2}{mr^3} = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

$$e^2 m r^3 = 4\pi\varepsilon_0 n^2\hbar^2 r^2$$

$$r = \frac{4\pi\varepsilon_0 n^2\hbar^2}{me^2}$$

$$= \frac{\varepsilon_0 h^2}{\pi m e^2} n^2. \quad (3)$$

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5(b). The total energy of the electron is $\epsilon = \epsilon_K + \epsilon_P$

$$= \tfrac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r}$$

again with the centripetal force $\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}$ multiply both sides by $\frac{r}{2}$.

$$\frac{1}{2}mv^2 = \frac{e^2}{8\pi\varepsilon_0 r}.$$

So $\epsilon = \frac{e^2}{8\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0 r}$.

$= -\frac{e^2}{8\pi\varepsilon_0 r}$ substitute in (3)

$= -\frac{e^2}{8\pi\varepsilon_0}\frac{\pi m e^2}{\varepsilon_0 h^2 n^2}$

$= -\frac{me^4}{8\varepsilon_0^2 h^2 n^2}$.

The ground state will be $n=1$ $\therefore$ $r = \frac{\varepsilon_0 h^2}{\pi m e^2}$ & $E = \frac{-me^4}{8\varepsilon_0^2 h^2}$.

(c). The four quantum numbers needed to describe the state of an electron are n, l, m_l and m_s. These quantum numbers are:

n: principal quantum number - it determines the radius of the electron orbit around the atom, and its binding energy to the atom. $\epsilon_n = \frac{-13.56}{n^2}$ eV

l: orbital quantum number - it gives the magnitude of the orbital angular momentum of the electron $|\underset{\sim}{L}| = \hbar\sqrt{l(l+1)}$

m_l: magnetic or azimuthal quantum number - tells us the orientation of $\underset{\sim}{L}$ relative to some direction z (eg. as defined by a weak magnetic field), so that the $\underset{\sim}{L}$ component along z is $L_z = m_l\hbar$

m_s: spin quantum number - tells us the orientation of the spin angular momentum $\underset{\sim}{S}$ relative to some direction z, so $S_z = m_s\hbar$.

Diagrams here, for m_l & m_s in particular, might be helpful but shouldn't be needed to give a complete & correct answer.

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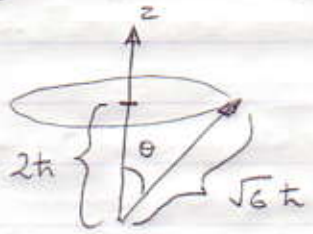
5(d). Fourth electron shell is $n = 4$. Allowed values of l are up to $n-1$, so $l = 0, 1, 2, 3$. Allowed values of m_l are integers between $\pm l$, so $m_l = -3, -2, -1, 0, 1, 2, 3$. Allowed values of m_s are $\pm 1/2$ as the electron is a spin-$1/2$ particle.

(e). In the Stern-Gerlach experiment a beam of neutral Ag atoms is passed through a non-uniform magnetic field which splits it into 2 beams, but the number of m_l states $= 2l+1$, which is odd indicating that there must be another quantum number. If the experiment is repeated with H instead of Ag, then $n=1$, $l=0$ and so $m_l = 0$ so there should only be one beam, but again two are observed. The experiment proves the existence of spin, the two beams being due to opposite deflection of the $m_s = \pm 1/2$ states.

(f). if the system has $l = 2$, then $m_l = -2, -1, 0, 1, 2$ and the smallest angle between $\underset{\sim}{L}$ and the z axis will be given by the largest magnitude of $m_l = \pm 2$. For $l = 2$ $|\underset{\sim}{L}| = \sqrt{l(l+1)}\,\hbar = \sqrt{2\times 3}\,\hbar = \sqrt{6}\,\hbar$ and $m_l = 2$ then $|\underset{\sim}{L_z}| = m_l \hbar = 2\hbar$, so



$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) = 35.3^\circ$$