Time allowed - 3 hours.

Total number of questions - 5.

Total marks - 100.

Answer **all** questions.

The questions are **not** necessarily of equal value.

Answer Part 1 on Electromagnetism and Part 2 on the Standard Model in separate books.

All working must be shown.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

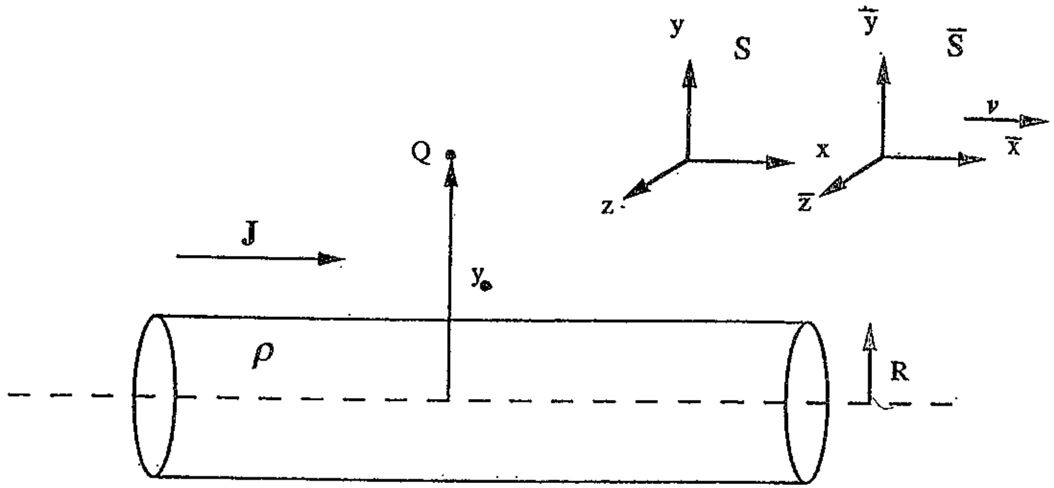
This paper may be retained by the candidate

Part 1 Electromagnetism

QUESTION 1 (19 marks)

Consider a charge Q at rest in reference frame S a perpendicular distance y_o from the centre of a stationary cylinder carrying a current density $J=(J_x,0,0)$ and with a static uniform net charge density $\rho = kJ_x$ where k is a constant. The cylinder is very long and is of circular cross-section with radius R.

Reference frame $\bar{S}$ has rectangular axes parallel with S. Reference frame $\bar{S}$ is moving with velocity $v = v\hat{x}$ relative to S along the common $x,\bar{x}$-axis. The magnitude of velocity v of $\bar{S}$ is $v = kc^2$. Assume that $v << c$ so that the non-relativistic 3-force F is equal to the relativistic 3-force $\mathcal{F}$.



Two physicists, one in S and the other in $\bar{S}$ know only Coulomb's Law and the force law $F = QE$ and relativistic mechanics.

(a) Using Coulomb's Law the physicist in S calculates that the non-relativistic 3-force F on the charge Q due to the charged cylinder is

(v) Show that if another physicist in frame $\bar{S}$ also knows Ampère's Law and calculates that the magnetic field $\bar{\boldsymbol{B}}$ in $\bar{S}$ at the charge is

$$\bar{\boldsymbol{B}} = \frac{\mu_o \bar{I}}{2\pi y_o} \hat{z}$$

where $\bar{I}$ is the current in $\bar{S}$, and uses the Lorentz force law s/he obtains the same result for $\bar{\boldsymbol{F}}$ as in part (a)(iii).

QUESTION 2 (6 marks)

Determine whether $g_{\mu\nu}$, the "metric coefficients" of Lorentz space, form the components of a tensor of this space. You must show all working and reasoning.

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QUESTION 3 (25 marks)

For a particle with charge q and mass m moving with 4-velocity $\eta^\mu = \gamma_u(c, \boldsymbol{u})$, and hence 4-momentum $\mathfrak{p}^\mu = m\eta^\mu$, moving in an applied electromagnetic field given by $F^{\mu\nu}$ on the data sheet, page 6, the Lorentz 4-force law is

$$K^\mu = \frac{d\mathfrak{p}^\mu}{d\tau} = qF^{\mu\nu}\eta_\nu \tag{1}$$

where τ is the proper time variable for the charged particle.

(a) Show from equation (1) with $\mu = 0$ that the power absorbed by the particle is equal to the rate of work done on it by the applied electric field

$$\frac{d\mathcal{E}}{dt} = q\boldsymbol{E}.\boldsymbol{u} \tag{2}$$

where the $\mathcal{E}$ is the relativistic energy of the charged particle, $\mathcal{E} = \gamma_u mc^2$.

(b) The covariant form for the total instantaneous power P_{rel} radiated by the charged particle is

$$P_{\text{rel}} = \frac{1}{4\pi\epsilon_o}\frac{2}{3}\frac{q^2}{m^2c^3}\left(\frac{d\mathfrak{p}_\mu}{d\tau}\frac{d\mathfrak{p}^\mu}{d\tau}\right) \tag{3}$$

(i) By evaluating $\mathfrak{p}^\mu\mathfrak{p}_\mu$, show that

$$\mathcal{E}^2/c^2 - |\mathfrak{p}|^2 = m^2c^2 \tag{4}$$

where $\mathfrak{p}$ is the relativistic 3-momentum.

(ii) Differentiate equation (4) with respect to proper time and show that

$$\frac{d\mathcal{E}}{d\tau} = |\boldsymbol{u}|\frac{d|\mathfrak{p}|}{d\tau}. \tag{5}$$

(iii) Hence show that

$$\frac{d\mathfrak{p}_\mu}{d\tau}\frac{d\mathfrak{p}^\mu}{d\tau} = \frac{d\mathfrak{p}}{d\tau}\cdot\frac{d\mathfrak{p}}{d\tau} - \frac{u^2}{c^2}\left(\frac{d|\mathfrak{p}|}{d\tau}\right)^2. \tag{6}$$

Consider the charged particle being accelerated to relativistic speeds in a linear accelerator where the only component of the momentum is in the direction $\hat{\boldsymbol{x}}$ of the applied electric field of magnitude E.

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Standard Model

Exam 2006

The two questions below have same value.

Question 4. Phenomenology.

a. Name all known leptons and quarks.
b. Present values for electric charges of leptons and quarks in the first generation.
   Hint: remember that the absolute value of the charge satisfies

$$|Q| = 1, \frac{1}{3}, \frac{2}{3} \tag{0.1}$$

(in units of the electron charge).

Consider a naïve, simplified physical picture for hadrons, in which they are presumed to be constructed from a minimal possible number of quarks ("Lego-type" model).

c. How many quarks are necessary to build the proton p ? Name them.
d. How many quarks are necessary to construct a pion π^- ? Name them.
   Hint: remember that the proton is a fermion and pion is a boson, and that they are relatively light particles.
e. Draw simplest Feynman diagrams for the beta-decay of the muon

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu \tag{0.2}$$

f. Indicate whether the parity is preserved or violated in reaction (0.2). Give a *brief* physical reason to support your claim.

Question 5. Standard Model

Consider the Weinberg-Salam $U(1) \times SU(2)$ gauge theory of electro-weak interactions.

a. How many gauge bosons are their in this model ?

Following the Weinberg-Salam theory presume that a scalar field (Higgs field), which is transformed as a doublet under $SU(2)$ (isospin $1/2$) and is also transformed nontrivially under $U(1)$) develops the vacuum expectation value

Hint: this question is closely related to b.

d. Consider a term in the Lagrangian, which describes interactions of leptons (first generation) with gauge bosons of the $SU(2)$ gauge group

$$L_{\text{int}}^{SU(2)} = \frac{g_2}{2}\left(\bar{L}\gamma^{\mu} W_{\mu}^{a} \tau^{a} L\right) \tag{0.5}$$

where L is a doublet of leptons

$$L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix}, \tag{0.6}$$

Extract from the Lagrangian (0.5) the terms, which are responsible for interaction of leptons with charged bosons $W^{\pm}$. Remember that

$$W_{\mu}^{\pm} = \frac{W_{\mu}^{1} \mp i W_{\mu}^{2}}{\sqrt{2}} \tag{0.7}$$

and the Pauli matrixes in Eq.(0.5) read

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.8}$$

Hint: the necessary terms in the Lagrangian are proportional to either the current $\left(\bar{\nu}_e \gamma_{\mu} e_L\right)$ or the current $\left(\bar{e}_L \gamma_{\mu} \nu_e\right)$.

e. Explain why an attempt to take the scalar field (Higgs field) as an isovector (isospin 1, i. e. triplet under $SU(2)$), which develops the vacuum expectation value

$$\phi_{\text{vac}} = \begin{pmatrix} 0 \\ 0 \\ \text{v} \end{pmatrix} \tag{0.9}$$

does not comply with experimental data.
Hint: the answer is related to

i. the family of gauge transformations, which leave the vacuum expectation value in Eq.(0.9) unchanged
ii. and the spectrum of masses of gauge bosons, which are responsible for the weak interaction

Gradient operator $\nabla = \hat{\boldsymbol{x}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\,\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

Gradient of scalar field ϕ $\nabla\phi = \frac{\partial \phi}{\partial x}\,\hat{\boldsymbol{x}} + \frac{\partial \phi}{\partial y}\,\hat{\boldsymbol{y}} + \frac{\partial \phi}{\partial z}\,\hat{\boldsymbol{z}}$

Divergence of vector field $\boldsymbol{v}$ $\nabla.\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

Curl of vector field $\boldsymbol{v}$

$$\nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

Second derivatives
Cartesian orthogonal coordinates

Laplacian operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Laplacian of scalar field ϕ $\nabla^2\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

Laplacian of vector field $\boldsymbol{v}$ $\nabla^2\boldsymbol{v} = \nabla^2 v_x\,\hat{\boldsymbol{x}} + \nabla^2 v_y\,\hat{\boldsymbol{y}} + \nabla^2 v_z\,\hat{\boldsymbol{z}}$

Electromagnetism in 3 D Euclidean space and time

$$\nabla.\boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\mathcal{F} = q(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$$

$$\nabla.\boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla.\boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

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$$\boldsymbol{E} = -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$$

$$\boldsymbol{\nabla}.\boldsymbol{A} = -\mu_0\epsilon_0 \frac{\partial V}{\partial t} \qquad \text{(Lorentz gauge condition)}$$

$$\nabla^2 V - \mu_0\epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 \boldsymbol{A} - \mu_0\epsilon_0 \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J}$$

$$V \to V - \frac{\partial \lambda}{\partial t} \qquad \text{(Gauge transformation)}$$

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\lambda$$

Cross product of 3 vectors in Euclidean space

$$\boldsymbol{A} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Relation between 4 D coordinates in (flat) Lorentz spacetime and 3 D Euclidean space and time coordinates

$$x^\mu = (x^0,\, x^1,\, x^2,\, x^3) = (ct,\, x,\, y,\, z)$$

$$x_\mu = (x_0,\, x_1,\, x_2,\, x_3) = (-ct,\, x,\, y,\, z)$$

Lorentz spacetime interval

$$ds^2 = dx^\mu dx_\mu$$

Metric of 4 D Lorentz spacetime

Coefficients

$$g_{\mu\nu} = g^{\mu\nu} = \begin{cases} 0 & \text{for} \quad \mu \neq \nu \\ -1 & \text{for} \quad \mu = \nu = 0 \\ 1 & \text{for} \quad \mu = \nu \neq 0 \end{cases}$$

Matrix notation

Relation between covariant and contravariant components of a 4-tensor in S in Lorentz spacetime

Implied summation notation

$$H_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} H^{\rho\sigma}$$
$$H^{\mu}{}_{\nu} = g_{\nu\lambda} H^{\mu\lambda}$$
$$H_{\mu}{}^{\nu} = g_{\mu\lambda} H^{\lambda\nu}$$

Matrix notation

$$[H_{\mu\nu}] = [g_{\mu\rho}][H^{\rho\sigma}][g_{\nu\sigma}]^{\mathrm{T}}$$
$$[H^{\mu}{}_{\nu}] = [H^{\mu\lambda}][g_{\nu\lambda}]^{\mathrm{T}} \qquad \text{where T indicates transpose}$$
$$[H_{\mu}{}^{\nu}] = [g_{\mu\lambda}][H^{\lambda\nu}]$$

Covariant and contravariant components of 4-gradient operator in Lorentz spacetime

4-gradient operator

$$\frac{\partial}{\partial x^{\mu}} \equiv \partial_{\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\frac{\partial}{\partial x_{\mu}} \equiv \partial^{\mu} = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

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Derivatives of a 4-vector field in Lorentz spacetime

First derivatives

$$\frac{\partial A^\mu}{\partial x^\nu} \equiv \partial_\nu A^\mu$$

$$\frac{\partial A^\mu}{\partial x_\nu} \equiv \partial^\nu A^\mu$$

$$\frac{\partial}{\partial x^\mu} \equiv g_{\mu\nu}\frac{\partial}{\partial x_\nu} \qquad \text{or} \qquad \partial_\mu = g_{\mu\nu}\partial^\nu$$

$$\frac{\partial}{\partial x_\mu} \equiv g^{\mu\nu}\frac{\partial}{\partial x^\nu} \qquad \text{or} \qquad \partial^\mu = g^{\mu\nu}\partial_\nu$$

Second derivatives

$$\frac{\partial}{\partial x_\nu}\frac{\partial A^\mu}{\partial x^\nu} = \partial^\nu\partial_\nu A^\mu$$

Scalar products of two 4-tensors in Lorentz spacetime (Contraction on two indices)

$$A^{\mu\nu}B_{\mu\nu} = A_{\mu\nu}B^{\mu\nu} = \mathrm{tr}\left\{[A_{\mu\nu}]^{\mathrm{T}}[B^{\mu\nu}]\right\}$$ where tr indicates trace and T indicates transpose

Scalar product of two 4-vectors in Lorentz spacetime (Contraction on one index)

$$a^\mu b_\mu = a_\mu b^\mu = [a_\mu]^{\mathrm{T}}[b^\mu]$$ where T indicates transpose

Scalar product of a 4-vector and a 4-tensor in Lorentz spacetime (Contraction on one inde:

$$a^\mu A_{\mu\nu} = b_\nu$$

$$[a^\mu]^{\mathrm{T}}[A_{\mu\nu}] = [b_\nu]$$ where T indicates transpose

Derivatives of a 4-tensor field in Lorentz spacetime

$$\frac{\partial H^{\mu\nu}}{\partial x^\nu} \equiv \partial_\nu H^{\mu\nu}$$

$$\frac{\partial H^{\mu\nu}}{\partial x_\nu} \equiv \partial^\nu H^{\mu\nu}$$

$$\alpha^\mu = \frac{d\eta^\mu}{d\tau} = \gamma_u \left[c\frac{d\gamma_u}{dt}, \frac{d(\gamma_u \boldsymbol{u})}{dt} \right]$$

4-force vector

$$K^\mu = \gamma_u \left[\frac{\boldsymbol{u}.\boldsymbol{\mathcal{F}}}{c}, \boldsymbol{\mathcal{F}} \right]$$

$$\boldsymbol{\mathcal{F}} = \frac{d\mathfrak{p}}{dt} = \frac{d(\gamma_u \boldsymbol{p})}{dt}$$

$$\boldsymbol{p} = m\boldsymbol{u}$$

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

4-energy-momentum vector

$$\mathfrak{p}^\mu = \left[\frac{\mathcal{E}}{c}, \mathfrak{p} \right]$$

$$\mathfrak{p} = \gamma_u m \boldsymbol{u}$$

$$\mathcal{E} = \gamma_u m c^2$$

4-charge-current density vector

$$J^\mu = \rho_0 \eta^\mu = [\rho c, \rho \boldsymbol{u}] = [\rho c, \boldsymbol{J}] = \rho_0 \gamma_u (c, \boldsymbol{u})$$

4-electromagnetic potential vector

$$A^\mu = \left[\frac{V}{c}, \boldsymbol{A} \right]$$

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A 4-vector in Lorentz spacetime transforms from S to $\overline{S}$ as defined previously according to

Implied summation notation

$$a^{\overline{\mu}} = \Lambda^{\overline{\mu}}{}_{\nu} a^{\nu}$$

Matrix notation

$$\left[a^{\overline{\mu}}\right] = \left[\Lambda^{\overline{\mu}}{}_{\nu}\right]\left[a^{\nu}\right]$$

where

$$\Lambda^{\overline{\mu}}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is the Lorentz transformation matrix and $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v/c$.

A 4-tensor in Lorentz spacetime transforms from S to $\overline{S}$ according to

Implied summation notation

$$H^{\overline{\mu}\overline{\nu}} = \Lambda^{\overline{\mu}}{}_{\lambda}\Lambda^{\overline{\nu}}{}_{\rho} H^{\lambda\rho}$$

Matrix notation

$$\left[H^{\overline{\mu}\overline{\nu}}\right] = \left[\Lambda^{\overline{\mu}}{}_{\lambda}\right]\left[H^{\lambda\rho}\right]\left[\Lambda^{\overline{\nu}}{}_{\rho}\right]^{T} \qquad \text{where T indicates transpose}$$

Covariant formulation of electromagnetism

$$F^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{array} \right\}$$

$$K^{\mu} = qF^{\mu\nu}\eta_{\nu}$$

$$\partial_{\mu}J^{\mu} = 0$$