Electrodynamics and Standard Model for Honors
Exam 2009

Please answer all 6 questions. If math proves complicated explain the idea in English.

Part I. Electrodynamics

Question 1 (Lorenz invariance, 20%)

a. Prove that the interval $\Delta s^2 = c^2\Delta t^2 - dr^2$ is invariant under the Lorenz transformations [1]

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \qquad (1.1)$$

in which v is the relative velocity (along the x- axes) between the reference frames K' and K.

b. Prove that the derivatives $\frac{\partial \phi}{\partial x^\mu}, \mu = 0,...3$ of a scalar function $\phi(x)$ transform under the Lorenz transformation as components of a covariant vector.
Hint. Remember that

$$x^\mu = (ct, \mathbf{r}) \qquad (1.2)$$
$$x_\mu = (ct, -\mathbf{r})$$

are the covariant and contravariant vectors respectively. To answer the question it suffices to consider any example of a scalar function, so that one can take the simplest (nonconstant) $\phi(x)$ that one can imagine.

c. Consider the action for a free particle $S = \varepsilon t - \mathbf{p}\cdot\mathbf{r}$, where ε is its energy, and $\mathbf{p}$ is the momentum. Presume that the action is a scalar. Derive a relation between the energy ε, momentum $\mathbf{p}$ and the mass of the particle m. Verify that this expression is consistent with the following ones

$$\varepsilon = \frac{mc^2}{\sqrt{1 - v^2/c^2}}, \quad \mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} \qquad (1.3)$$

where v is the velocity. Hint: use Q1b combining it with the nonrelativistic expression between the energy, momentum and mass.

d. Consider an object, which moves with the velocity v in relation to the observer.

- Present simple qualitative arguments demonstrating that the size of the object in the direction perpendicular to the velocity remains same for any velocity.
- Derive an expression, which describes "shortening" of the object in the direction of motion. Hint: use Eqs.(1.1) (and be careful with the definition of the length).

e. Derive an expression, which describes the "slowing down" of time in a spacecraft from the perspective of an observer on Earth. The spacecraft moves with the velocity v in relation to the Earth. Hint: use Eqs.(1.1) (and be careful with the definition of the period of time).

Question 2 (Equations of motion, 20%)

---

[1] If it is convenient, the system of units with c=1 can be used for answering all the questions.

a. Consider the potential A^μ. Remember that it is a four-vector $A^\mu = (A_0, \mathbf{A})$. Express the field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ via the electric and magnetic fields **E** and **B**.
Hint: It is convenient to fulfil this task step by step as follows

- Calculate F_{01}. Then immediately, without calculations, one can write down the answer for $F_{0m} = -F_{m0}$, m=1,2,3.
- Calculate F_{12}. Then without further calculations one can write F_{mn}, m,n = 1,2,3.

b. Prove that the equation

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} j^\nu \tag{2.1}$$

where $j^\mu = (c\rho, \mathbf{j})$ is the four-current, represents the Gauss' law for the electric field and the Ampere's law.

c. Consider a particle with mass m and charge e, which moves in the external electromagnetic field. Remember that the equation of motion for this particle reads

$$mc\frac{du^\mu}{ds} = \frac{e}{c} F^{\mu\nu} u_\nu \tag{2.2}$$

where $u^\mu = (1,\, \mathbf{v}/c)\dfrac{1}{\sqrt{1-v^2/c^2}}$ is the four-velocity. Prove that in the three-dimensional notation the equation of motion reads

$$\frac{d\mathbf{p}}{dt} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right) \tag{2.3}$$

d. Consider a particle with the mass m and charge e, which moves in the homogeneous static magnetic field **B** in the plane perpendicular to the field.

- Prove that the particle moves over a circular trajectory.
- Find the frequency of rotation of the particle along this trajectory.
- Express this frequency via the energy of the particle ε, its mass, charge and the magnetic field.
- Find the ratio of the relativistic frequency to the known cyclotron frequency (which represents the nonrelativistic limit).

Question 3 (Fields, potentials, radiation, 10%)

a. Consider the gauge transformation $A_\mu \to A_\mu' = A_\mu + \partial_\mu f$. Prove that the field $F_{\mu\nu}$ remains gauge invariant.

b. Consider the rate of energy radiated by a charged particle which propagates in the electromagnetic field. The relativistic expression for it reads

$$d\varepsilon = \frac{2e^2}{3m^2c^3}\frac{1}{1-\mathbf{v}^2/c^2}\left[\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right)^2 - \frac{1}{c^2}(\mathbf{E}\cdot\mathbf{v})^2\right]dt \tag{3.1}$$

Derive from this the known nonrelativistic limit for the rate of radiated energy.

c. Consider a particle in the magnetic field presuming that conditions of Q2d are satisfied (homogeneous, static magnetic field, velocity is perpendicular to the field).

- Calculate the rate of the energy radiated by this particle.
- Indicate how the rate depends on the energy of the particle ε.

- Estimate the period of time during which the particle radiates a substantial amount of its initial energy ε presuming that $\varepsilon \gg mc^2$.

Part II. Standard Model

Question 4 (phenomenology, 10 %)

a. Write down the gauge group, which governs the Standard Model
- Explain which part of this group is responsible for the electromagnetic and weak interactions
- Which group governs the strong interactions

b. Name all the forces, which this symmetry embraces

c. Name the gauge bosons responsible for these forces
- For each boson state its spin and its mass

d. Describe briefly the scale of distances for each of the forces
- Explain briefly why the weak interactions in nuclei have a short radius
- Explain briefly how confinement manifests itself for strong interactions
- Explain briefly which phenomenon is called the asymptotic freedom

e. Indicate how many generations of leptons and quarks are known, name all leptons and quarks in each generation, indicate the charges of leptons and quarks.
Hint: remember that the absolute values of the charge satisfies

$$|Q|=1, \frac{1}{3}, \frac{2}{3}$$

(in units of the electron charge).

f. Consider a naïve, simplified physical picture for hadrons, in which they are presumed to be constructed from a minimal possible number of quarks ("Lego-type" model).
- How many quarks are necessary to build the proton p ? Name them.
- How many quarks are necessary to construct a pion π^- ? Name them.

Hint: remember that the proton is a fermion and pion is a boson, and that they are relatively light particles.

g. Present the Feynman diagram that describes the beta-decay of the neutron into the proton and leptons. Indicate whether the parity is preserved or violated in this reaction. Give a brief physical reason to support your claim.

Question 5 (gauge theory, 30 %)

Consider a gauge transformation \

$$\begin{aligned} &\psi \rightarrow \psi' = U\psi \\ &A_\mu \rightarrow A_\mu' = UA_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} \end{aligned} \tag{5.1}$$

where ψ is a wave function, which is a n-dimension complex column, A_μ is a Hermitian traceless $n\times n$ matrix, and U is a $n\times n$ unitary unimodular matrix, $U^{-1}=U^\dagger$, $\det U=1$.

a. Verify that the covariant derivative $\nabla_\mu=\partial_\mu-igA_\mu$ is transformed as

$$\nabla_\mu\psi\to\nabla_\mu'\psi'=U\nabla_\mu\psi \tag{5.2}$$

b. Consider the gauge field, which is defined as

$$F_{\mu\nu}=\frac{i}{g}[\nabla_\mu,\nabla_\nu] \tag{5.3}$$

Prove that

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-ig[A_\mu,A_\nu] \tag{5.4}$$

c. Consider the *SU*(2) gauge group. Present the vector potential A_μ via an isotopic vector $A_\mu^a, a=1,2,3$ using the following relation

$$A_\mu=\frac{1}{2}A_\mu^a\tau^a \tag{5.5}$$

- Show that the field $F_{\mu\nu}$ can also be presented as an isotopic vector, call it $F_{\mu\nu}^a$, which satisfies

$$F_{\mu\nu}=\frac{1}{2}F_{\mu\nu}^a\tau^a \tag{5.6}$$

- Show that this vector satisfies

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\,\varepsilon^{abc}A_\mu^b A_\mu^c \tag{5.7}$$

Hint: use Eq.(5.4) and remember that the Pauli matrixes satisfy $[\tau^a,\tau^b]=2i\,\varepsilon^{abc}\tau^c$.

d. Explain briefly, qualitatively how $F_{\mu\nu}^a$ are transformed under the gauge transformations. (Hint: the answers is simple.)

e. Explain briefly, qualitatively how $F_{\mu\nu}^a F^{a\mu\nu}$ are transformed under

- gauge transformations
- Lorentz transformations

Question 6 (Higgs mechanism, 20 %)

a. Give a brief qualitative description of the phenomenon known as the Higgs mechanism. Explain its relation to the masses of *W*, and *Z* bosons.

b. Explain briefly an advantage of the Higgs isotopic doublet compared to an isotopic triplet.

Hint: consider the mass of the *Z* boson in a theory, in which the Higgs field is an isotopic triplet.

c. Describe the Higgs mechanism in detail for the photon, which propagates in the superconductor presuming that the superconductring properties are due to the condensate of Cooper's pairs..