THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

EXAMINATION – MAY 2007

PHYS4103 – PHYS IV (HONOURS)
UNIT A QUANTUM MECHANICS

Time Allowed – 3 hours
Total Number of Questions – 4
All questions are NOT OF EQUAL value
All questions should be attempted
Answer Questions 1 and 2 in one exam book.
Answer Questions 3 and 4 in the second exam book.
Candidates may bring their own calculators.
Answers must be written in ink. Except where they
are expressly required, pencils may only be used
for drawing, sketching or graphical work

Honours Quantum 2007

Question 1

Consider ground state of a Helium atom. Both electrons are in the 1s orbital state with the wave function $\varphi_{1s}(r)$. You do not need an explicit form of $\varphi_{1s}(r)$.
A) Write down the two-electron wave function including both the orbital and the spin part. Explain why spin of the ground state is zero.

Consider excited Helium atom. One of the electrons is in 1s state, and the second electron is in the 2s state. The orbital wave functions of these states are $\varphi_{1s}(r)$ and $\varphi_{2s}(r)$. The total spin in this case can be zero, $S = 0$, or one, $S = 1$.
B) Write down two-electron wave functions in both cases. Show both the orbital and the spin part of the wave functions.

Account for Coulomb interaction between electrons

$$V(1,2) = \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

by perturbation theory and hence calculate the corresponding energy shift like expectation value of the interaction over the wave function.
C) Calculate energy shifts for states with $S = 0$ and $S = 1$ using wave functions found in the part **B)**. Express your answers in terms of integrals and clearly indicate which integral is called the "direct interaction" and which integral is called the "exchange interaction".
D) What is the energy splitting between the state with $S = 0$ and the state with $S = 1$? Express your answer in terms of the above integrals.

Question 2

The Hamiltonian of a charged particle in static magnetic field is

$$H = \frac{(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{2m} .$$

Consider a charged particle in an uniform magnetic field directed along the z-axis, $\mathbf{B} = (0, 0, B)$, and use a gauge with vector potential $\mathbf{A} = (0, Bx, 0)$.
A) Check that this vector potential corresponds to the magnetic field.
B) Perform separation of variables in Schroedinger equation and find the wave function of the particle in a stationary quantum state.
C) Find the spectrum of the system (Landau levels).
You may use without proof the spectrum of 1D harmonic oscillator, $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \to \epsilon = \hbar\omega\left(n + \frac{1}{2}\right)$.

Quantum Mechanics part II
Exam 2007

Questions 3 and 4 have values 20 and 30 (from a total of 50 for part II).

Question 3. Relativistic equations.

i. Write down the Dirac equation for charged fermions in an external electromagnetic field.
ii. Write down the fundamental algebraic relation, which defines the Dirac matrixes γ^μ, $\mu = 0,1,2,3$.
iii. Solve the Dirac equation for a free fermion with zero momentum $\mathbf{p} = 0$.
Hint: this task is simplified, if the following representation for the Dirac matrixes is used

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{3.1}$$

- How many energy levels are there?
- Are they degenerate or not ? Why ?
- What is called the Dirac sea ?
- Why is it necessary to presume that fermions satisfy the Dirac statistics?

iv. Prove that a matrix γ_5, which is defined

$$\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{3.2}$$

anticommutes with any Dirac matrix γ^μ, $\mu = 0,1,2,3$, i.e. that

$$\gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu \tag{3.3}$$

Hint: it is almost a one-liner, if the fundamental relation for the Dirac matrixes (see ii) is applied.

Question 4. Scattering problem.
Consider elastic scattering of a nonrelativistic particle by a spherically symmetric potential $U(r)$.

i. Perturbation theory

a. Give a definition of the scattering amplitude $f(\theta)$.
b. Explain qualitatively what is called the first Born approximation.
c. Explain how the first Born approximation is related to a general formulae for the scattering amplitude, which reads

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int \exp(-i\mathbf{k}'\cdot\mathbf{r})U(r)\psi_{\mathbf{k}}(\mathbf{r})d^3r \tag{4.1}$$

where $\mathbf{k}, \mathbf{k}'$ are the initial and the final wave vectors.

d. Assume that the potential has the form,

$$U(r) = \frac{C}{r}\exp(-\mu r) \tag{4.2}$$

where C, μ are two constants.
Argue that the Born approximation is applicable in this problem, when either C is small, or μ is large.

Assuming that the Born approximation is applicable

- Calculate the scattering amplitude as a function of energy E of the incoming particle and scattering angle θ.
- Calculate the scattering length.
- Calculate the differential cross section
- Write down an expression for the total cross section (but do not waist your time calculating the integral in it explicitly for arbitrary energies).
- However, calculate the total cross section for zero energy.

Hint: you will need the formula

$$\int \frac{1}{r}\exp(-\mu r + i\mathbf{q}\cdot\mathbf{r})d^3r = \frac{4\pi}{q^2+\mu^2} \tag{4.3}$$

ii. Scattering phases.

a. Calculate the scattering phases $\delta_l(k)$ as functions of the orbital momentum l and the momentum $k = \sqrt{2mE}$ of the incoming particle in the potential

$$U(r) = \frac{\beta}{2mr^2} \tag{4.4}$$

where β is a constant, and m is a mass of the particle (which is introduced in Eq.(4.4) to simplify the formulas).

b. Prove that the potential (4.4) must not be too attractive, i.e. β must satisfy condition

$$\beta \geq -\beta_0. \tag{4.5}$$

Find the constant $\beta_0 > 0$.

Hints. Keep in mind:

1. The wave function $R_{kl}(r)$ in the l-th partial wave at large distances behaves as

$$R_{kl}(r) \to \frac{2}{r}\sin\left(kr - l\pi/2 + \delta_l(k)\right),\ r \to \infty \tag{4.6}$$

2. The Laplacian for the l-th partial wave reads

$$\Delta = \Delta_r - \frac{l(l+1)}{r^2} \tag{4.7}$$

where $\Delta_r = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$ is its radial part.

3. Eq.(4.7) implies that for the given potential (4.4) the Schrödinger equation can be solved analytically (introducing an effective orbital momentum L).