Honours Quantum Mechanics 2016
Part I, 50/100 marks

Question 1

A (**12 marks**) Explain origin/mechanism of the exchange interaction between electrons. In the explanation you can use any example.

B (**3 marks**) Consider Helium-like, Beryllium-like and Carbon-like ions of Ge. The ions consist of Germanium nucleus and two, four, and six electrons respectively. Ground state electronic configurations of these ions are

$$He-like: 1s^2$$
$$Be-like: 1s^2 2s^2$$
$$C-like: 1s^2 2s^2 2p^2$$

Determine spins of ground states of these ions.

C (**10 marks**) Explain microscopically why values of spins are such as you present in your answer to the question **B**, hence explain the Hund's rule.

Question 2

Electron dynamics in graphene-like two-dimensional (2D) quantum material are described by the following Hamiltonian

$$H = v(\vec{\sigma} \cdot \vec{p}) + 2\eta \sigma_z s_z \ ,$$

where v is the Fermi-Dirac velocity, η is the strength of the spin-orbit interaction, $\vec{\sigma}$ describes sublattice pseudospin and $\mathbf{s}$ is the usual spin of the electron. Both pseudospin and spin are described by Pauli matrices acting in different spaces, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

Orbital dynamics are 2D, therefore the momentum is $\vec{p} = (-i\partial_x, -i\partial_y)$.

A (12 marks). Consider a plane wave state of the electron,

$$\psi = e^{ik_x x + ik_y y} \chi_\sigma \chi_s \ ,$$

where χ_σ is pseudospin spinor and χ_s is spin spinor. Solve Dirac equation, $H\psi = \epsilon\psi$, and hence
(i) Find the energy of the state and sketch the electron dispersion.
(ii) Determine direction of the pseudospin and direction of the spin in the positive-energy state.
(iii) Determine direction of the pseudospin and direction of the spin in the negative-energy state.

B (13 marks). Magnetic field B perpendicular to the 2D plane is applied to system The corresponding Dirac Hamiltonian reads

$$H = v(\vec{\sigma} \cdot [\vec{p} - e\vec{A}]) + 2\eta \sigma_z s_z - 2B s_z \ .$$

Solve Dirac equation and hence find Landau levels in this system.
Hint: It is convenient to use a planar gauge, $\mathbf{A} = (0, Bx, 0)$ *or* $\mathbf{A} = (-By, 0, 0)$.

$(-i)^2 = i^2 = -$

$(iP_y)(-iP_y)$

$-i^2 P_y^2$

Exam 2016
Quantum Mechanics for honours
Part II

Address both question II.1 and II.2; each of them contributes 25% to the total mark.

The system of units $\hbar = 1$ is used throughout. Additionally, in the relativistic problem II.1 the condition $c = 1$ is also taken.

Question II.1 Relativistic equations (25%)

A. Consider the potential well

$$e\,A_0(\mathbf{r}) = U(\mathbf{r}) = \begin{cases} -V, & r < a \\ 0, & r \geq a \end{cases} \tag{1}$$

A-a. Wright down explicitly the Klein - Gordon equation for the scalar $\phi(\mathbf{r})$, which describes propagation of a scalar particle with mass m in this potential.

A-b. Find an explicit transcendential equation, which governs the energy spectrum of bound states in the S-wave.

Hint: write down the solution of the Klein-Gordon equation inside and outside of the potential and compare them at the boundary $r = a$, where the logarithmic derivative

$$\left(\frac{1}{r\,\psi(r)} \frac{d\,(r\,\psi(r))}{d\,r} \right)_{r=a} \tag{2}$$

should be equal for both of them.

Compare the found equation with the one, which follows from the Schrodinger equation in the nonrelativistic case. Estimate the region of parameters V, a, m, in which the relativistic correction proves to be essential.

B. Using the Dirac equation and the spinor representation for the Dirac matrixes,

should have $-I$?

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix} \tag{3}$$

prove that the energy spectrum of free fermions necessarily incorporates two branchers, the positive $\varepsilon > m$ and negatives $\varepsilon < -m$ energies.

Question II.2 Scattering problem (25%)

A. In the first Born approximation find the differential cross section $\frac{d\sigma}{d\Omega}(\theta)$ for the scattering of the particle of mass m and energy E on the potential

$$U(r) = \frac{\gamma}{r^2} \tag{4}$$

Prove that total cross section is infinite, $\sigma = \infty$.

Present an estimate for the parameter γ, which guarantees that the first Born approximation is applicable.

Hint: remember that

$$\int \frac{e^{i\, q \cdot r}}{r^2}\, d^3 r = \frac{1}{4\pi q} \tag{5}$$

B. Assume that the particle of mass m propagates in the field of the spherically symmetric repulsive potential

$$U(r) = \begin{cases} W, & r < a \\ 0, & r \geq a \end{cases} \tag{6}$$

Here W and a are two parameters, $W > 0$ measures the intensity of the repulsion and a is the radius of the sphere inside which the potential operates. Assume that W is large, $W \to \infty$.

B-a. Find the wave function $\psi(r)$ for the scattering problem at zero energy, $E = 0$. Present the scatch of the function $(r\,\psi(r))$ versus r; compare it with the case when scattering is negligible ($W \to 0$).

B-b. Using the wave function found derive the expression for the scattering length λ; present also the cross section σ at zero energy.

Consider also the behaviour of the same particle in the field of another potential. Take the attractive potential which is localized within the sphere of a small radius b; presume that in this potential there exists a shellow, S-wave bound state with the energy $E_b = -\kappa^2 / (2\, m)$, where κ is so small that $\kappa\, b \ll 1$ (in this case the potential is called the "zero radius" one).

B-c. Find the wave function, $\psi_{E=0}(r)$, for the scattering problem at zero energy, $E = 0$, in the region outside the potential, $r \geq b$.

Hint: express this function in terms of κ. This can be achieved if one remembers that the condition on the logarithmic derivative

$$\left(\frac{1}{r\,\psi(r)} \frac{d(r\,\psi(r))}{d r} \right)_{r \simeq b \approx 0} \tag{7}$$

shows no substantial variation under a small variation of the energy considered. Consequently, the condition can be firstly derived for the bound state problem (for which the wave function in the outside region is known). After that the same condition can be used for the wave function of the scattering problem.

B-d. Find the scattering length λ and cross section σ for the scattering on the potential of zero radius. State what happens with these parameters when the binding becomes very weak, $\kappa \to 0$.

B-e. Presesent the scatch of the found wave function $(r\,\psi_{E=0}(r))$ versus r. Compare it with the wave function found for the repulsive potential in question **B-a**. State whether the behaviour of the two functions outside of the potentials, i.e. for $r > a$ or b, is qualitatively similar or differrent.

B-f. The combined findings from **B-a-e** allow one to address the question whether the *attractive* potential can exhibit *repulsive* influence on the particle located outside of this potential. State your view on this point supporting it by your findings from **a-e.**