THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

MAY 2011

PHYSICS HONOURS

Quantum Mechanics

Time Allowed – 3 hours

Total number of questions - 4

Answer ALL questions

All questions are of equal value

Answer Part I and Part II in separate booklets

Candidates may bring their own calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

Candidates may keep this paper.

Part One

Question 1

Consider H_2 molecule consisting of two electrons and two protons.
You might need the following data.
The electron mass is $m_e \approx 0.511$MeV, the proton mass is $m_p \approx 938$MeV, the fine structure constant is $\alpha \approx 1/137$. The average distance between protons in the ground state of the H_2 molecule is $r_0 \approx 0.74\mathring{A} \approx 1.4a_B$. To help with numerical calculations I remind that the atomic unit of energy is $\frac{\hbar^2}{m_e a_B^2} = 27.2eV$.

A. Draw schematically spectrum of the energy levels, including electronic, vibrational, and rotational levels.

B. Write down the wave function of the molecule in the adiabatic approximation.

C. Based on Schrodinger equation estimate
(i) the typical electronic energy,
(ii) the typical vibrational energy,
(iii) the typical rotational energy.
Present your answers in parametric form in terms of Rydberg and ratios of electron and proton masses and also give numerical values in electron volts.

D. Based on Schrodinger equation estimate
(i) the typical momentum of electron,
(ii) the typical momentum of proton.
Present your answers in units of inverse Bohr radius.

E. Spin of a proton is $1/2$, so the total nuclear spin of the molecule can be $S_N = 0$ or $S_N = 1$. What is the total nuclear spin of the molecule in the ground state? Explain your answer, do not forget about Fermi statistics for protons. Calculate the energy splitting between the lowest $S_N = 1$ state (orthohydrogen) and the lowest $S_N = 0$ state (parahydrogen). Present a formula for the splitting and give also the answer in eV. This splitting is widely used in neutron physics.

Question 2

Consider a quantum wire in which an electron can move freely in the y-direction while its motion in the x- and z-directions is restricted by the potential

$$U(x,z) = \frac{m\omega^2(x^2+z^2)}{2},$$

where m is the electron mass and ω is the parameter of the transverse confining potential. A uniform magnetic field $\mathbf{B} = (0,0,B)$ directed along the z-axis is applied to the system. *Use the gauge $A = (0, Bx, 0)$ for the vector potential.*

A. Write down the Hamiltonian of an electron in the wire.

B. Using separation of variables, $\psi(x,y,z) = e^{ipy/\hbar}\varphi(z)\chi(x)$, solve the Schrodinger equation and find energy levels of the electron. Notice that the motion along the wire is described by the plane wave.

The wire is filled by electrons with linear density ρ. The system is in the ground state (do not forget the Fermi statistics). Assume that the frequency of the confining potential ω is sufficiently large such that only the lowest transverse level is occupied. Assume also that electrons are spinless, in other words disregards all effects related to spin.

C. Calculate the energy of the system per unit length.

Hint: The energy has a term linear in ρ and another term cubic in ρ.

You may use without proof the spectrum of 1D harmonic oscillator,
$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \to \epsilon = \hbar\omega\left(n + \frac{1}{2}\right)$.

$$\frac{(p_x + eB_y)^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

Quantum Mechanics for Honours, Part II
Exam 2011

Total mark for Part II is 50, questions **1** and **2** give 25 each.

Question 1 (Part II).
Relativistic equations (feel free to use absolute units or relativistic units with $\hbar = c = 1$)

1. Write down the Dirac equation for the spinor ψ:
    - For free fermions
    - Write down the fundamental algebraic relation, which defines the Dirac matrixes γ^{μ}. Explain why this relation is necessary.
    - Write down the Dirac equation in an external electromagnetic field. Explain what is called the gauge invariance. Present explicitly gauge transformations for the potential and the wave function.
    - Present the Dirac equation in the external field in the form of the Schrödinger-type equation, in which the matrixes $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$, $\beta = \gamma^0$ are used.

2. Solve the Dirac equation for a free electron with momentum $\mathbf{p} = 0$. With this purpose
    - Find explicitly the energies and wave functions of this problem.
    Hint: assume that the solution can be written in the form
    $$\psi = u \exp\left(\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - \varepsilon t)\right), \tag{1.1}$$
    where the Dirac spinor u does not depend on the coordinates and reduce the Dirac equation to the algebraic equation on u. This task is simplified if the following representation (standard representation) of the Dirac matrixes is taken
    $$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{1.2}$$
    - Calculate the z-projections of spin for the solutions you find.
    Hint: remember that the operator of the spin z-projection reads
    $$S_z = \frac{1}{2}\Sigma_z = \frac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.3}$$
    - Explain the physical meaning of all solutions you find.

3. Solve the Dirac equation for free fermions in the ultrarelativistic limit. In order to do this assume that the solution can be written in the form of Eq.(I.1)

Consider the case of large momentum and high energy $p \gg mc$, $|\varepsilon| \gg mc^2$ (small mass case) when the mass term is irrelevant (the chiral limit). Find explicit expressions for the available energy levels ε and the corresponding spinors u.

- Indicate how many energy levels are there for a given momentum.
- Are they degenerate or not ? Why ?
- Find the z-projections of spin for these solutions.
- Explain which solutions have left and which ones have right chirality.

Hint: this task is simplified if the following representation (spinor representation, which is *different* from the standard representation (I.1)) for the Dirac matrixes is used

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{1.4}$$

the particle is presumed to propagate along the z-direction, and a conventional representation for the Pauli matrix σ_z is taken

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

4. Write down the Klein-Gordon equation for a charged scalar particle propagating in an attractive Coulomb field which creates the potential energy $U = -\dfrac{Z\alpha}{r}$.

   - Prove that the discrete energy levels for this particle read

$$\varepsilon_{n,l} = mc^2 \left(1 + \frac{Z^2\alpha^2}{\left(\sqrt{(l+1/2)^2 - Z^2\alpha^2} - l - 1/2 + n\right)^2} \right)^{-1/2} \tag{1.6}$$

   Here $l = 0, 1\ldots$, $n = 1, \ldots$

   Hint: introduce the effective energy E and effective (non-integer) orbital momentum L in such a way that the Klein-Gordon equation reveals a similarity with the Schrödinger equation for the Coulomb problem. Then apply the Rydberg formula

$$E = -mc^2 \frac{Z^2\alpha^2}{2n^2} \tag{1.7}$$

   - Find restriction on Z in this problem. Outline (very briefly) possible physical implications of this restriction.

Question 2 (Part II).
Scattering problem. Consider elastic scattering of a nonrelativistic particle with mass m on a spherically symmetrical potential $U(r)$ (feel free to use absolute units or units with $\hbar = 1$)

1. Describe qualitatively (*very briefly*) how the perturbation theory is formulated for the scattering problem on the potential. In particular:

   - Explain qualitatively (briefly) what is called the scattering amplitude $f(\theta)$.
   - Indicate how the differential and total cross sections $\frac{d\sigma}{d\Omega}$, σ are expressed via the amplitude $f(\theta)$.

2. Using the Lippmann-Schwinger equation

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} U(r)\psi_{\mathbf{k}}(\mathbf{r}')d^3r' \qquad (2.1)$$

   prove that the scattering amplitude reads

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{k}'\cdot\mathbf{r}}U(r)\psi_{\mathbf{k}}(\mathbf{r})d^3r \qquad (2.2)$$

   Hint: at large r the approximation

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{e^{ikr - i\mathbf{k}'\cdot\mathbf{r}'}}{r}, \qquad \mathbf{k}' = k\frac{\mathbf{r}}{r} \qquad (2.3)$$

   is applicable.

3. Simplify the amplitude in Eq.(2.2) when the Born approximation is applicable.

4. Consider scattering of a particle with the mass m on the potential well

$$U = \begin{cases} -U_0, & r \le a \\ 0, & r > a \end{cases} \qquad (2.4)$$

   - Find the scattering amplitude at low energy in the first Born approximation.
     Hint: low energy allows one to presume that $ka \ll 1$, where a is the size of region where the potential is located.
   - Present condition, which guarantees that the first Born approximation is applicable in this case.
     Hint: if you do not remember this condition by hart, you can derive it by stating that a typical kinetic energy in the vicinity of the potential should be higher than the potential energy.

5. Assume that the scattering phase δ_0 (s-phase) exhibits the following behaviour

$$\cot\delta_0 = -\frac{Q}{k} \tag{2.5}$$

where k is the wave vector ($\hbar k$ is the momentum) of the incoming particle and Q is a given parameter, which characterizes the potential and is small (i.e. much smaller than any other similar parameter for this potential).

- Find the scattering amplitude at low energy when $k \sim |Q|$.
- Find the total cross section explicitly.
- Assuming that Q is positive present a (brief) argument, which assures that there exists a discrete energy level for the particle in the given potential with low binding energy B; find this B.

Hint. Remember the general relation between the scattering amplitude and phases

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\frac{\exp(2i\delta_l)-1}{2ik}, \tag{2.6}$$

which at low energy reads

$$f \approx \frac{\exp(2i\delta_0)-1}{2ik} = \frac{1}{k}e^{i\delta_0}\sin\delta_0 \tag{2.7}$$

(Please keep in mind that the final answers in this problem are very simple, do not try anything that looks complicated.)

6. Present the total cross section σ via the scattering phases (at arbitrary energy). Hint: use Eq.(2.6) and the known identity

$$\int P_l(\cos\theta)P_{l'}(\cos\theta)d\Omega = \frac{4\pi}{2l+1}\delta_{l,l'} \tag{2.8}$$

7. Consider the low-energy scattering, when the s-wave gives the dominant contribution. Express the cross section via the phase δ_0 (applying, e.g. Eq.(2.7)). Assume that the potential can be tuned in such a way that δ_0 can take any value.

- By tuning the potential (i.e. tuning the phase δ_0) find the particular situation when the cross section σ reaches its maximum for the given momentum $\hbar k$. Find the explicit values for δ_0 and σ in this case.
- By tuning the potential find the situation when the cross section σ reaches its minimum for the given momentum $\hbar k$. Present the explicit values for δ_0 and σ in this case.

Suggestion: if there are difficulties suggest simple physical arguments, which would still allow one to find the maximum and minimum of the cross section.