Question 1

Consider ground state of a Helium atom. Both electrons are in the 1s orbital state with the wave function $\varphi_{1s}(r)$. You do not need an explicit form of $\varphi_{1s}(r)$.
A) Write down the two-electron wave function including both the orbital and the spin part. Explain why spin of the ground state is zero.

Consider excited Helium atom. One of the electrons is in 1s state, and the second electron is in the 2s state. The orbital wave functions of these states are $\varphi_{1s}(r)$ and $\varphi_{2s}(r)$. The total spin in this case can be zero, $S = 0$, or one, $S = 1$.
B) Write down two-electron wave functions in both cases. Show both the orbital and the spin part of the wave functions.

Account for Coulomb interaction between electrons

$$V(1,2) = \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

by perturbation theory and hence calculate the corresponding energy shift like expectation value of the interaction.
C) Calculate energy shifts for states with $S = 0$ and $S = 1$ using wave functions found in the part (B). Express your answers in terms of integrals and clearly indicate which integral is called the "direct interaction" and which integral is called the "exchange interaction".

Question 2

The Hamiltonian of a charged particle in static magnetic field is

$$H = \frac{(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{2m} .$$

Consider a charged particle in an uniform magnetic field directed along the z-axis, $\mathbf{B} = (0, 0, B)$, and use a gauge with vector potential $\mathbf{A} = (-By, 0, 0)$.
A) Check that this vector potential corresponds to the magnetic field.
B) Perform separation of variables in Schroedinger equation and find the wave function of the particle in a stationary quantum state.
C) Find the spectrum of the system (Landau levels).
You may use without proof the spectrum of 1D harmonic oscillator, $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \rightarrow \epsilon = \hbar\omega\left(n + \frac{1}{2}\right)$.
D) Assume that the charged particle is confined in a box with dimensions $L_x \times L_y \times L_z$. So the cross section of the box perpendicular to the magnetic filed has the area $S = L_x L_y$. Show that the degeneracy of a given Landau level is

$$\Delta N = \frac{|e|BS}{2\pi\hbar c} .$$

Quantum mechanics, Part II, Exam 2009

Please answer all questions, if math is difficult, give a descriptive answer. One is welcomed to use either the absolute units, or the system of units, in which the Plank constant and the velocity of light are unities ($\hbar = c = 1$) for all answers.

Q1 (Part II) Relativistic equations (25%)

A. Free Dirac particle

- Write down the Dirac equation for free electrons
- Write down the fundamental relation, which defines the properties of the Dirac matrixes γ^μ. Explain why this condition is necessary.
- Take the spinor particle with mass zero, $m = 0$. Consider the following representation for the Dirac matrixes

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \tag{1}$$

Write the Dirac spinor ψ in the following form

$$\psi = \begin{pmatrix} R \\ L \end{pmatrix} \tag{2}$$

where R and L are two-component spinors, which means

$$R = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}, \qquad L = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

Verify that the Dirac equation for ψ results in two independent equations for R and L.

- Find the spectrum for the states described by the spinor L, i.e. find the energy ε as a function of momentum $-\infty < p_z < \infty$ and projection of spin $s_z = \pm 1/2$. Verify that the energy is positive provided L has left chirality (which means that the momentum p_z and the z-projection of spin have the opposite signs).

  Hint: consider the stationary equation on the spinor L, which follows from the Dirac equation when Eqs. (1) and (2) are applied.
- Explain the physical meaning of all those solutions, in which the state L has negative energies.

B. Electron in external field

- Write down the Dirac equation for an electron, which propagates in an external electromagnetic field described by the four-potential A_μ.
- Consider particular case, an electron propagating in a homogeneous, constant magnetic field **B** aligned along the z-axes. Write down the Dirac equation for the stationary state with energy ε and zero projection of the momentum on the axes along the field, $p_z = 0$.

  Hint: this means that the spinor $\psi(\mathbf{r}, t)$ reads

$$\psi(\mathbf{r}, t) = e^{-i\varepsilon t}\psi(x, y)$$

- Presume further that the field $\mathbf{B}$ is so strong that the electron mass is negligible in the problem, $m = 0$. Rewrite the Dirac equation in a form of the two equations for two-component spinors R and L introduced in Eq.(2).
- Take the equation for L and rewrite it further in a form of the second-order differential equation, which reads

$$\varepsilon^2 L = \hat{H}^2 L. \quad (3)$$

  Here $\hat{H}^2$ is a second-order differential operator, which you need to find.

  Hint: the first-order differential equation incorporates the operator $\boldsymbol{\sigma}\cdot(\hat{\mathbf{p}} - e\mathbf{A}/c)$, where $\hat{\mathbf{p}} = -i\hbar\nabla$ and $\mathbf{A}$ are the operator of momentum and the vector potential respectively. In order to derive the second-order differential equation one needs to consider the square of this operator $[\boldsymbol{\sigma}\cdot(\hat{\mathbf{p}} - e\mathbf{A}/c)]^2$. Remember that the latter can be conveniently written as follows

$$\left(\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)\right)^2 = \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 - \frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\mathbf{B} = \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 - \frac{e\hbar}{c}B\sigma_z$$
$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

  (An advantage of the second-order differential equation is that only diagonal 2×2 matrixes are present there, as you should see.)
- Find the spectrum of states for the Dirac particle in the homogeneous magnetic field. i.e. find the energy levels ε from Eq.(3).

  Hint. The equation, which you derived in (3) should be close in form to the Schrödinger equation, which reads

$$E\psi = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 \psi \quad (5)$$

  Remember also the formula for the Landau levels, which follows from the Schrödinger equation (5)

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_B, \qquad \omega_B = \frac{|e|B}{mc}, \qquad n = 0, 1\ldots \quad 6)$$

  Using it one can derive the relativistic result without sophisticated calculations.

Q2 (Part II) Scattering problem (25%)

Consider a non-relativistic particle, which has the momentum $\hbar\mathbf{k}$ and mass m and propagates in the spherically symmetrical potential *U(r).*

A. Definitions

- Formulate briefly definition for the scattering amplitude $f(\theta)$ via the asymptotic behaviour of the wave function $\psi_{\mathbf{k}}(\mathbf{r})$, which describes the scattering problem.
- Present an expression for the differential cross section $\frac{d\sigma}{d\Omega}$ via the scattering amplitude (do not derive it).
- Prove that the amplitude can be written as follows

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{k}'\cdot\mathbf{r}}\, U(r)\psi_{\mathbf{k}}(\mathbf{r})\, d^3r \qquad (7)$$

Here **k** and **k'** are the wave vectors of the incoming and outgoing particles.

Hint: start from the Schrödinger equation and remember the formulae, which allows one to resolve the differential equation on $f(\mathbf{r})$

$$\left(\Delta + k^2\right) f(\mathbf{r}) = Q(\mathbf{r}) \qquad (8)$$

via the integral

$$\psi(\mathbf{r}) = -\frac{1}{4\pi}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} Q(\mathbf{r}')\, d^3r' \qquad (9)$$

Note also an approximate equality

$$|\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{n}\cdot\mathbf{r}', \quad \mathbf{n} = \mathbf{r}/r, \quad \text{when} \quad r \to \infty \qquad (10)$$

B. Born approximation

- Explain very briefly how the perturbation theory for the scattering problem is formulated, what is called the Born approximation.
- Prove that in the Born approximation the following identity holds

$$f(\theta) = -\frac{m}{2\pi\hbar^2} V(\mathbf{q}) \qquad (11)$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ and $V(\mathbf{q})$ is the Fourier component of the potential

$$V(\mathbf{q}) = \int U(\mathbf{r})\, e^{i\mathbf{q}\cdot\mathbf{r}} d^3r \qquad (12)$$

Hint: Eq.(7) may be useful.

- Consider the Coulomb potential

$$U(r) = -\frac{\alpha}{r} \qquad (13)$$

  a. Using the Born approximation find the scattering amplitude $f(\theta)$.
  Hint: remember that

$$\int \frac{1}{r} e^{i\mathbf{q}\cdot\mathbf{r}}\, d^3r = \frac{4\pi}{\mathbf{q}^2} \qquad (14)$$

  b. Find the differential cross section $\frac{d\sigma}{d\Omega}$ in the first Born approximation presenting it via the energy of the particle $E = \frac{\hbar^2 k^2}{2m}$, the scattering angle θ and constant α introduced in Eq.(13).

C. Low energy scattering, resonance at low energy, partial wave expansion

- Explain briefly
  a. what is called the scattering length λ,
  b. why the scattering length does not depend on the scattering angle θ and why is it real
- Presume that for some atom there exists the state of a negative ion with zero binding energy. Consider scattering of an electron on this neutral atom.
  a. Find the cross section for low-energy electron-atom scattering. Show explicitly how the cross section depends on energy.
  b. Find the scattering phase $\delta_0(k)$ for $l = 0$ and low energy.