THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

EXAMINATION – MAY 2006

PHYS4103 – PHYS IV (HONOURS)
UNIT A QUANTUM MECHANICS

Time Allowed – 3 hours

Total Number of Questions – 4

All questions are of EQUAL value

All questions should be attempted

Answer Questions 1 and 2 in one exam book.

Answer Questions 3 and 4 in the second exam book.

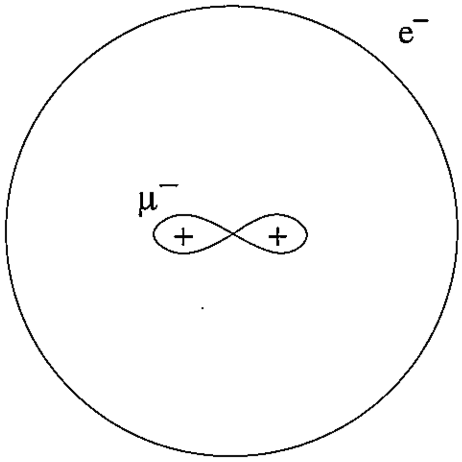
Candidates may bring their own calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

This paper may be retained by the candidate

Question 1

Consider a neutral system that consists of two protons, a negative muon, and an electron. The protons are bound by the muon into a $pp\mu$ molecule. The electron is orbiting around the molecule. The mass of the muon is $m_\mu \approx 200 m_e$ and the mass of the proton is $m_p \approx 2000 m_e$,



where m_e is the electron mass. The atomic unit of energy is $\frac{m_e e^4}{\hbar^2} \approx 27.2 eV$ and the atomic unit of length is $a_B = \frac{\hbar^2}{m_e e^2} \approx 0.53\ 10^{-8} cm$. Disregard spins and magnetic interactions of the particles. Estimate parametrically and numerically the following quantities

1) Size of $pp\mu$ molecule.
2) Size of the entire system.
3) Dissociation energy of $pp\mu$ molecule.
4) Vibrational energy of $pp\mu$ molecule.
5) Rotational energy of $pp\mu$ molecule.
6) Compare these energies with the excitation energy of the electron and hence draw schematically the spectrum of the system.

Question 2

The Hamiltonian of a charged particle in static magnetic field is

$$H = \frac{(\mathbf{p} - \frac{e}{c}\mathbf{A})^2}{2m} .$$

Consider a charged particle in an uniform magnetic field directed along the z-axis, $\mathbf{B} = (0, 0, B)$, and use a gauge with vector potential $\mathbf{A} = (-By, 0, 0)$.

1) Check that this vector potential corresponds to the magnetic field.
2) Perform separation of variables in Schroedinger equation and find the wave function of the particle in a stationary quantum state.
3) Find the spectrum of the system (Landau levels).

You may use without proof the spectrum of 1D harmonic oscillator, $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2} \rightarrow \epsilon = \hbar\omega\left(n + \frac{1}{2}\right)$.

Quantum Mechanics part II
Exam 2006

Questions 3,4 have same value

Question 3. Dirac equation.

I. Write down the Dirac equation for the spinor ψ:
   - For free fermions
   - For charged fermions in an external electromagnetic field
   - Write down the fundamental algebraic relation, which defines the Dirac matrixes γ^{μ}.

II. Solve the Dirac equation for free fermions in the ultrarelativistic limit. In order to do this assume that the solution can be written in the form

$$\psi = u \exp\left(\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r} - \varepsilon t)\right), \tag{3.1}$$

where the Dirac spinor u does not depend on the coordinates. Consider a large momentum and energy $p \gg mc, |\varepsilon| \gg mc^2$, when the mass term becomes irrelevant in the equation (chiral limit), and find explicit expressions for the available energy levels ε and the corresponding spinors u.
   - How many energy levels are there for a given momentum?
   - Are they degenerate or not ? Why ?

Hint: this task is simplified if the following representation for the Dirac matrixes is used

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \tag{3.2}$$

the particle is presumed to propagate along the z-direction, and a conventional representation for the Pauli matrix σ_z is taken

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

III. Prove the current conservation law for free fermions

$$\partial_{\mu} j^{\mu} = 0, \tag{3.4}$$

where the current is defined as follows

$$j^{\mu} = (c\rho, \mathbf{j}), \quad \rho = \psi^{+}\psi, \quad \mathbf{j} = c\psi^{+}\boldsymbol{\alpha}\psi. \tag{3.5}$$

where ψ^{+} means the Hermitian conjugation of ψ, i.e.

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \psi^{+} = \begin{pmatrix} \psi_1^*, & \psi_2^* & \psi_3^* & \psi_4^* \end{pmatrix} \tag{3.6}$$

and the derivatives have conventional meaning

$$\partial_{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{3.7}$$

Hint: a simple way provides the Schrödinger form of the Dirac equation. Write it down and use twice, for ψ and ψ^+. (Feel free to use any other way of derivation, if it looks more attractive for you.)

Question 4. Scattering problem. Consider elastic scattering of a nonrelativistic particle by a spherically symmetrical potential $U(r)$.

I. Describe *briefly*, qualitatively how the perturbation theory is formulated for the scattering problem. In particular:

- Explain what is called the first Born approximation.
- Present condition, which guarantees that the first Born approximation is applicable for low impact energies.

Hint: if you do not remember this condition, you can derive it by stating that a typical kinetic energy in the vicinity of the potential should be higher than the potential energy.

II. Consider representation of the scattering amplitude via the scattering phases

$$f(\theta)=\sum_l (2l+1)P_l(\cos\theta)\frac{\exp(2i\delta_l)-1}{2ik} \tag{4.1}$$

- Using Eq. (4.1)derive the optical theorem, which related $\operatorname{Im} f(0)$ with the total cross section σ.

Hint: remember that the Legendre polynomials satisfy the following condition

$$\int P_l(\cos\theta)P_{l'}(\cos\theta)d\Omega=\frac{4\pi}{2l+1}\delta_{l,l'} \tag{4.2}$$

III. Assume that the s-phase exhibits the following behaviour

$$\cot\delta_0=-\frac{\kappa}{k} \tag{4.3}$$

where k is the momentum of the incoming particle and κ is a parameter, which proves be small compared with any other (similar) parameter that characterizes the potential).

- Derive from Eqs.(4.1),(4.3) an explicit expression for the scattering amplitude
- Derive from Eqs.(4.1),(4.3) an explicit expression for the differential $\frac{d\sigma}{d\Omega}(\cos\theta)$ cross section; indicate in which direction θ this cross section exhibits the maximum (if there is one)
- Find the total σ cross section

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