Quantum Field Theory
Exam for honours students,
School of Physics UNSW, Sep 2014

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1. Allocated time 3 hours.
2. Each of the four questions below should be addressed.
3. Each one has the same value 25 %.
4. If math is a problem, try to find physical arguments.
5. Relativistic units $\hbar = c = 1$, $e^2 = \alpha \approx 1/137$ are used for formulation of the problems, you can rely on the same set of units in your replies, or use any other, say SI units, stating the point clearly.
6. The calculators are allowed, but may not be extremely helpful.

Question 1. Vacuum polarization

1. Explain very briefly, qualitatively the phenomenon called the vacuum polarization:
   a. discuss its physical origins and manifestations,
   b. outline basic properties of the Uehling potential
   c. explain which phenomenon is called the Landau pole
   d. explain briefly why polarization of vector particles produces the asymptotic freedom, not the Landau pole

μ - vacuum:

2. Consider a nucleus of charge Z, which creates the Coulomb potential energy for the electron

$$U_C = -\frac{Z\alpha}{r} \qquad (1)$$

$\frac{1}{\sqrt{a_B^3}} e^{-r/a_B}$

Remember the expression for the Uehling potential energy in this case

$$U_{\text{Uehl}}(r) = -\frac{Z\alpha}{r}\left(1 + \frac{2\alpha}{3\pi}\int_1^\infty e^{-2mr\zeta} f(\zeta)\, d\zeta\right)$$

$\delta E_{\text{Uehling}} = \langle 1s | U(r) | 1s \rangle$

$\frac{2\alpha}{3\pi}\int_1^\infty e^{-2mr\zeta}\frac{\sqrt{\zeta^2-1}}{\zeta^2}\left(1+\frac{1}{2\zeta^2}\right)d\zeta$

where $\int^\infty \frac{f(\zeta)}{\zeta^2} = \frac{2}{5}$

$$f(\zeta) = \frac{\sqrt{\zeta^2 - 1}}{\zeta^2}\left(1 + \frac{1}{2\zeta^2}\right) \qquad (3)$$

a. Give an estimate for the energy shift due to the vacuum polarization for an atomic $1s$ electron (an accurate numerical coefficient is not required here, keep only important physical parameters). Similarly consider the energy shift of 1s atomic electron, which arises due to the vacuum polarization of the muon vacuum. Expain why the former (energy shift due to electron polarization) is larger than the latter (energy shift due to muon polarization), present the parameter which governs this outcome.

b. Consider electron-nuclear collision. Calculate the lowest (in powers of α) correction produced by the vacuum polarization to the differential cross section. Remember that

$$\frac{d\sigma}{d\Omega} = |f|^2 \qquad (4)$$

$\langle 1s | U | 1s \rangle$

where f is the scattering amplitude. Assuming that the electron velocity v is nonrelativistic but large

$$1 > v > Z\alpha \qquad (5)$$

calculate the amplitude in the nonrelativistic Born approximation when it reads

$$f \approx f_B = -\frac{m}{2\pi}\int e^{-i q \cdot r}\, U(r)\, d^3 r \qquad (6)$$

where q is the transfered momentum, which is expressed via the initial and final electron momenta as follows

$$q = p_i - p_f \qquad (7)$$

Hints

- It is convenient to present the answer in the form

(8)

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Coul}} (1 + \alpha\, X(\theta)) \qquad (9)$$

where $X(\theta)$ needs to be found as a function of the scattering angle θ (turns out to be quite simple function).

- A couple of useful formulas:

$$\int e^{i k \cdot r}\frac{e^{-\mu r}}{r} d^3 r = \frac{4\pi}{k^2 + \mu^2} \qquad (10)$$

$a_B = \frac{1}{\alpha m}$

$v > Z\alpha$.

$\int_1^\infty \frac{f(\zeta)}{\zeta^2}$

$\frac{v^2}{m^2}$

$V_{0,N} = \langle 0|\hat V|N\rangle = \sum_N \langle 0|\hat p|N\rangle \frac{1}{m}$

$= \frac{1}{m}\sum_N [\langle 0|\hat p|N\rangle$

$$\int_1^\infty \frac{f(\zeta)}{\zeta^2}\,d\zeta = \frac{2}{5}$$

$\frac{1}{m}\langle 0|HF|N\rangle$

$\frac{1}{m}\sum_N\{\frac{1}{2}\langle 0|\hat p|N\rangle\langle N|[H,\hat p]|0\rangle - \langle\ \rangle\langle\ \rangle\}$

$\langle 0|\hat p[H,\hat r]|N\rangle \ldots$

Question 2. Lamb shift

1 Explain briefly, in simple physical terms the phenomenon called the Lamb effect.

In particular outline

- its manifestations
- those physical reasons which justify its presence,
- the role played by the vacuum polarization and self - energy

2 Derive the following "starting" formula for the nonrelativistic part of the self - energy correction to the Lamb shift

$$\delta E_{ns} = e^2 \int \frac{d^3k}{(2\pi)^3} \sum_{e_k}\sum_N \frac{2\pi}{\omega} \frac{|\langle N\,p\,|\,\mathbf{v}\cdot\mathbf{e}_k\,|\,n\,s\rangle|^2}{E_{ns} - E_{Np} - \omega} \tag{12}$$

Here ns is the initial s-state of the electron and Np is the intermediate p-state, E_{ns} and E_{Np} are their energies, $\mathbf{v}$ is the operator of the electron velocity, $\boldsymbol{k}$, $\omega = k$, and $\boldsymbol{e_k}$ are the photon momentum, energy and polarization

Clarify in particular the following points.

$\Gamma = \mathrm{Im}(\delta E_{ns})$

- Outline briefly the physical origins of the factor $\frac{2\pi}{\omega}$. ✓
- Present the Feynman diagram ✓

$\vec d = d_0\cos(\omega t)$

$\ddot{\vec d} = \omega^2 d$

- Using this diagram explain the presence of the numerator $e^2|\langle N\,p\,|\,\mathbf{v}\cdot\mathbf{e}_k\,|\,n\,s\rangle|^2$ and the energy denominator in Eq.(12).
- Using Eq.(12) derive an expression for the radiative width Γ_{ns} of an ns level

$\frac{dW}{dt} = \frac{2}{3}\frac{\ddot d^2}{c^3}$ $\quad W = \frac{\Gamma}{2}$

Hint Take into account the following well known from conventional quantum mechanics rule When the energy denominator $E_i - E_{\rm int}$ in the perturbation theory turns zero, it should be treated as follows

$$\frac{1}{E_i - E_{\rm int}} \Longrightarrow \frac{1}{E_i - E_{\rm int} + i\,0} = \mathrm{Re}\left(\frac{1}{E_i - E_{\rm int}}\right) - i\,\pi\,\delta(E_i - E_{\rm int}) \tag{13}$$

✓ **Question 3. Interaction with photons.**

Consider an electron impacting on the proton Using the Born approximation for the electron wavefunction and assuming E1-type for the radiative transition find the cross section for the electron radiative capture to the 1s level, i e the cross section for the following reaction

$$e + p \to H_{1s} + \gamma \tag{14}$$

Describe the behaviour of the cross section at small and large electron energy E, i.e. at $E \to 0$, and $E \gg \mathrm{Ry}$

Hints

- The simplest way is to express the cross section for the radiative capture $\sigma_{\rm rad.cap.}$ via the cross section of the photoionization $\sigma_{\rm photo}$, which describes the reversed process

$$H_{1s} + \gamma \to e + p \tag{15}$$

and which can be considered as well known (remember the assignment).

- Alternatively, one needs to fulfill the full set of the following calculations, which relies on the following
  - The Fermi gloden rule
  - The electron wave functions

$$\psi_i = e^{i\,p\cdot r},\quad \psi_f = \psi_{1s}(r) = \frac{1}{\sqrt{\pi a_B^3}}\,e^{-r/a_B} \tag{16}$$

  - The Fourier component of the 1s wave function

$a_B = \frac{1}{m\alpha}$

$E_f = \omega - Ry$ $\quad \omega \gg Ry$ $\quad \langle 0|\hat v\cdot\vec e|N\rangle$

$E_i = \frac{p^2}{2m_e} + \frac{p^2}{2m_p}$

$$\int e^{i k \cdot r} \psi_{1s}(r)\, d^3 r = \frac{8\sqrt{\pi a_B^3}}{\left(1+(k a_B)^2\right)^2}, \tag{17}$$

- The cross section equals the probability per the flux

$$\sigma = \frac{W}{\text{flux}} \tag{18}$$

- The factor $\frac{2\pi}{\omega}$ for the photon
- The summation rule for summation over polarizations in E1 transitions, 2/3 rule for the coefficient, which facilitates calculations.

Question 4. Heisenberg - Euler effect in magnetic field.

1. Outline briefly, in simple physical terms, the phenomenon called the Heisenberg - Euler effect. Explain which physical reasons justify its presence, and outline its manifestations.

2. Consider a hypothetical 2 D world, in which electrons can propagate over the x - y plane, while their motion along z direction is restricted (this world can be modeled by some external force, which strongly quantises the electron motion along z - direction).

Assume that there is the magnetic field B applied along the z - axis, which produces conventional Landau levels for the electron motion in the x - y plane.

Find the energy per unit surface $\frac{\partial E}{\partial S}$ which arises due to the Heisenberg - Euler effect in this case.

In particular find the asymptotic behaviour at small and large fields. Point out if there is a qualitative distinction in these asymptotes from the conventional 3 D case.

Hints

$b = \frac{eB}{m}$

The Landau spectrum for electrons reads

$$\varepsilon = \pm\left(m^2 + |e|\, B\,(2n+1+\sigma)\right)^{1/2}, \quad \sigma = \pm 1, \ n = 0, 1 \ldots \tag{19}$$

The necessary summation over all quantum states amounts to

$$\frac{eB}{2\pi} \sum_{\sigma=\pm 1} \sum_{n=0}^{\infty} \quad \int^{\infty} d \tag{20}$$

In calculations it is convenient first to take the derivative $\frac{\partial}{\partial(m^2)} \frac{\partial E}{\partial S}$ then use the identity

$$\frac{1}{\sqrt{X}} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{d\xi}{\sqrt{\xi}}\, e^{-\xi X} \tag{21}$$

$\xi =$

and after that recover $\frac{\partial E}{\partial S}$ from $\frac{\partial}{\partial(m^2)} \frac{\partial E}{\partial S}$

$X = \varepsilon^2 \rightarrow$

- A handy expansion reads

$$\coth x \approx \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \ldots, \ x \to 0 \tag{22}$$

$$\frac{1}{\sqrt{X}} = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{d\xi}{\sqrt{\xi}} e^{-\xi X}$$

ξ

$m^2 + 2neB$

$m^2 + 2eBn + eB + eB\sigma$

$m^2 +$

$\frac{v^2 m}{2} \quad \frac{1}{m\alpha}$

$\langle i | \vec{p} \cdot \vec{\varepsilon} | f \rangle$

$a_B = 1/m\alpha \quad Ry = \frac{1}{2}\alpha^2 m$

$= \frac{1}{2} a_B m^2 \alpha^3$