THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

MAY 2011

PHYSICS HONOURS

Electromagnetism and the Standard Model

Time Allowed – 3 hours

Total number of questions - 4

Answer ALL questions

All questions ARE NOT of equal value

Candidates may bring their own calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

Candidates may keep this paper.

Electrodynamics and Standard Model for Honours

Exam 2011

There are two questions related to electrodynamics, and two to the Standard Model. Please address all of them. If you have difficulties with math give explanations in simple physical terms in plain English. Calculators are allowed, but would hardly produce real help. The questions are formulated in relativistic units $\hbar = c = 1$, unless stated otherwise. Feel free to use either absolute or relativistic units.

Part I. Electrodynamics

Question 1. Lorentz invariance and motion of charged particles (25%)

Consider two reference frames K ("classroom") and K' ("spaceship"), presuming that K' moves in relation to K with the velocity v along the x-direction.

a. Write down the Lorentz transformation, which relates coordinates x',y',z',t' in K' with x,y,z,t in K. Write it in a matrix form

$$x^\mu = \Lambda^\mu_\nu x'^\nu , \tag{1.1}$$

presenting the matrix Λ^μ_ν explicitly.

b. Write down the transformation which relates the scalar V' and vector A' potentials in the reference frame K' with the potentials V and A in the reference frame K.

Hint: use the fact that the potentials represent the four-vector

$$A^\mu = (V, \mathbf{A}). \tag{1.2}$$

c. Assume that the potentials satisfy the Lorenz condition

$$\frac{\partial V'}{\partial t'} + \nabla' \cdot \mathbf{A}' = 0 \tag{1.3}$$

in the reference frame K', where ∇' refers to the three-vector of derivatives over the coordinates x',y',z'. Find the condition, which the potentials V and A satisfy in the reference frame K.

d. Derive expressions, which relate the energy and momentum of a relativistic particle with its velocity v.
Hint: remember that

$$\frac{dx^\mu}{ds} = \frac{1}{\sqrt{1-v^2}}(1, \mathbf{v}) \tag{1.4}$$

is a four-vector, in which x^μ are the four coordinates, $ds=\sqrt{dx^\mu dx_\mu}=dt\sqrt{1-v^2}$ is the variation of the interval along the trajectory, $\mathbf{v}$ is the velocity, $v=|\mathbf{v}|$. Compare the four-vector of the velocity with the four-vector of the momentum. One can take first the reference frame where the particle rests and then using the Lorentz invariance extend the result for an arbitrary reference frame.

e. Consider the charged particle of mass m and charge e in the external electromagnetic field. Assume that in the reference frame where the particle rests (at a given moment of time) the equation of motion is know, has a very conventional simple form

$$m\mathbf{a}=e\mathbf{E}. \tag{1.5}$$

- Prove that in an arbitrary reference frame Eq.(1.5) can be rewritten as follows

$$m\frac{du^\mu}{ds}=e\,F^{\mu\nu}u_\nu\,. \tag{1.6}$$

Here $u^\mu=\dfrac{dx^\mu}{ds}$ is the 4-velocity defined by Eq.(1.4) and $F^{\mu\nu}$ is the field tensor

$$F_{\mu\nu}=\begin{pmatrix}0 & E_x & E_y & E_z\\ -E_x & 0 & -B_z & B_y\\ -E_y & B_z & 0 & -B_x\\ -E_z & -B_y & B_x & 0\end{pmatrix},\quad F^{\mu\nu}=\begin{pmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{pmatrix}. \tag{1.7}$$

Hint: it is convenient firstly to go the opposite way, from (1.6) to (1.5), and then to claim that the Lorentz invariance guarantees that as a matter of fact (1.6) follows from (1.5).

- Prove that the components $\mu=1,2,3$ of Eq.(1.6) can be presented in conventional 3D notation as follows

$$\frac{d\mathbf{p}}{dt}=e\big(\mathbf{E}+\mathbf{v}\times\mathbf{B}\big). \tag{1.8}$$

where

$$\mathbf{p}=\frac{m\mathbf{v}}{\sqrt{1-v^2}}\,. \tag{1.9}$$

- Write explicitly, in 3D notation the $\mu=0$ component of Eq(1.7). State the physical meaning of this equation.

Question 2. Fields (25%)

a. Verify the following:

- The field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfies Eq.(1.7)
- The dual field $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}F_{\lambda\rho}$,where $\varepsilon^{\mu\nu\lambda\rho}$ is the anti-symmetric tensor with $\varepsilon^{0123} = -\varepsilon^{1023} = \ldots = 1$, equals

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{2.1}$$

It suffices to verify these identities for only one component of the electric and magnetic fields, for example for E_x, B_x.

b. Prove that

$$\mathbf{E}^2 - \mathbf{B}^2 \text{ and } \mathbf{E}\cdot\mathbf{B} \tag{2.2}$$

are Lorentz invariant.
Hint: consider the quantities $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$.

c. Prove that the equation

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad j^\nu = (\rho, \mathbf{j}) \tag{2.3}$$

reproduces the Gauss' and Ampere's laws.

d. Assume that there exists the electric field **E**' in the reference frame *K*' (see Q1a). Using the Lorentz transformation find the field **E** in the reference frame *K*.
Hint: remember that

$$F^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\lambda F'^{\rho\lambda} \tag{2.4}$$

where Λ^μ_ν is from (1.1). Please keep in mind that the necessary in (2.4) calculations are very simple.

e. A relativistic particle with the mass m and charge e propagates in an external electromagnetic field **E**, **B**. For the ultrarelativistic case $\gamma = 1/\sqrt{1-v^2} \gg 1$ find the force **F**, which the radiation produced by this particle inflicts on the particle itself. Indicate how this force depends on γ when

- The magnetic field is absent and the electric field is orthogonal to the velocity $\mathbf{E} \neq 0,\ \mathbf{E} \cdot \mathbf{v} = 0,\ \mathbf{B} = 0$
- The magnetic field is absent and the electric field is parallel to the velocity $\mathbf{E} \neq 0,\ \mathbf{E} \times \mathbf{v} = 0,\ \mathbf{B} = 0$

Hint: in order to calculate **F**

- consider the radiated momentum $\dfrac{d\mathbf{p}}{dt}$
- keep in mind that in the ultrarelativistic case the radiated momentum is *very* simply related to the radiated energy
- remember that the rate of radiated energy reads (absolute units)

$$\frac{d\varepsilon}{dt} = -\left(\frac{2}{3}\right)\frac{e^4}{m^2c^3}\frac{\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)^2 - \frac{1}{c^2}(\mathbf{v} \cdot \mathbf{E})^2}{\left(1 - v^2/c^2\right)} \tag{2.6}$$

where ε is the energy of the particle.

$F = \frac{kg\, m}{s\, s}$

$F = \frac{dp}{dt}$

Part II. Standard Model
Question 3. Phenomenology (20%)

a. Name all the forces described by the Standard Model and the one fundamental force, which is not covered by the Standard Model. Explain very briefly qualitatively the difficulty with this force. In particular:

   - Name all the gauge bosons, which mediate each of the forces in the Standard Model, indicate their electric charges and masses (accurate values are not necessary, just the scale)
   - Present the scale of distances, which is typical for each of these forces.

   Explain briefly, qualitatively
   - which phenomenon is called confinement,
   - which property of the gauge bosons defines the radius at which the weak interaction is effective,
   - the reasons which prompt the weak interaction to manifest itself weakly in nuclear physics

b. Name the known leptons and quarks, arranging them in generations. Indicate the scale of masses for leptons and quarks (exact values are not necessary, rough estimations or qualitative statement would suffice). Show the values for the electric charges of leptons and quarks in the first generation.

c. Consider a simplified physical picture for hadrons, in which they are constructed from a minimal possible number of quarks ("Lego-type" model).
   Describe in this picture the structure of pions $\pi^{\pm}$ and kaons $K^{\pm}$ (remember that kaons are particles that have a similarity with pions, but are "strange" particles because they include one s (or anti-s) quark.

d. Consider a simplified, qualitative model of the vacuum state in the quantum field theory, which describes fermions. Assume that for each fermion the vacuum state is described by the Dirac sea. i. e. that there is the Dirac sea for electrons e, another Dirac sea for u quarks, yet another for d etc. Find the total charge of all fermions in the vacuum state. With this purpose:
   - For each fermion take a state in its Dirac sea with one and the same (for all fermions) momentum $\mathbf{p}$ and same projection of spin σ.
   - Find the charge of all those fermions in the Dirac seas, which have this momentum $\mathbf{p}$ and projection of spin σ. Start from the first generation and then generalize your result to include other generations. (Hint: do not forget that each quark has three colours, so that for a given $\mathbf{p}$ and σ there are three different states for the quark.)
   - Integrating over all momenta $\mathbf{p}$ and spin projections σ one finds the total charge of the vacuum.
   - Argue that your result can be used to relate the values of the charges of quarks with the number of their colours.

e. Present the Feynman diagrams responsible for elastic scattering of the low-energy electron by the neutron. Outline which qualitative phenomena this process produces in atoms.

Question 4. Gauge theory (30%)

a. Consider the wave function ψ which describes some particle. Assume that it is an isotopic doublet $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$. Assume that the theory possesses the gauge symmetry SU(2) and this wave function is transformed by this symmetry.
Verify that the gauge transformation of the wave function

$$\psi \to \psi' = U\psi,\ U \in \mathrm{SU}(2) \tag{4.1}$$

leaves $\psi^+\psi$ invariant.

b. Remember that the vector potential A_μ in the SU(2) gauge theory is represented by a 2×2 Hermitian, traceless matrix, $A_\mu^+ = A_\mu$, $\mathrm{Tr}(A_\mu) = 0$, which means that $A_\mu \in su(2)$, where $su(2)$ is the gauge algebra. Remember also that for the wave function ψ the covariant derivative

$$\nabla_\mu = \partial_\mu + i\,g\,A_\mu \tag{4.2}$$

is defined as follows

$$\nabla_\mu \psi = \partial_\mu \psi + igA_\mu \psi \tag{4.3}$$

Prove that if we postulate that this derivative is transformed by the gauge transformations similarly to the way the wave function is transformed

$$\nabla_\mu \psi \to \nabla_\mu{}' \psi' = U \nabla_\mu \psi \tag{4.4}$$

(compare Eq.(4.1)) then the vector potential needs to be transformed as follows

$$A_\mu \to A'_\mu = UA_\mu U^{-1} + \frac{1}{ig} U \partial_\mu U^{-1} \tag{4.5}$$

c. Prove that $F_{\mu\nu}$ is a 2×2 Hermitian, traceless matrix (i.e. $F_{\mu\nu} \in su(2)$)
Hints:

- Remember the definition

$$F_{\mu\nu} = \frac{1}{ig}[\nabla_\mu, \nabla_\nu] \tag{4.6}$$

- Using (4.6) find explicit expression for the field $F_{\mu\nu}$ in terms of the potentials A_μ, and then using it verify the necessary identities

$$F_{\mu\nu} = F_{\mu\nu}^+,\ \mathrm{Tr}(F_{\mu\nu}) = 0 \tag{4.7}$$

d. Prove that under the gauge transformation the field is transformed as follows

$$F_{\mu\nu} \to F'_{\mu\nu} = UF_{\mu\nu}U^{-1} \tag{4.8}$$

Hint: rewrite firstly Eq.(4.3) as follows

$$\nabla_{\mu} \to \nabla'_{\mu} = U\nabla_{\mu}U^{-1} \tag{4.9}$$

and then use the definition of the field in (4.6).

e. Suppose that some object X is transformed by the gauge transformation as follows

$$X \to X' = UXU^{-1} \tag{4.10}$$

(which is different from the transformation of the wave function in (4.1) and similar to the field transformed in (4.8))
Then the covariant derivative for this object is defined as follows

$$\nabla_{\mu}(X) = \partial_{\mu}X + ig\left[A_{\mu}, X\right] \equiv \left[\nabla_{\mu}, X\right]. \tag{4.11}$$

- Prove that the derivative of X is transformed similarly to X itself, i.e. prove that

$$\nabla_{\mu}(X) \to \nabla_{\mu}'(X') = U\left(\nabla_{\mu}(X)\right)U^{-1}. \tag{4.12}$$

- Take the vector $J^{\mu} \in su(2)$, call it the current. Its explicit form at the moment is irrelevant. It is important only to know that under gauge transformations this vector exhibits the following variation

$$J_{\mu} \to U\,J_{\mu}\,U^{-1}, \tag{4.13}$$

which is similar to the one in Eqs.(4.8) and (4.10).

Prove that the equation

$$\nabla_{\mu}\left(F^{\mu\nu}\right) = J^{\nu} \tag{4.14}$$

is gauge invariant.

Congratulations, you have just shown that in the non-Abelian gauge theory one can formulate equations, which are similar in nature to Maxwell's equations in electrodynamics. This is remarkable.