THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

PHYS3550 - GENERAL RELATIVITY

FINAL EXAMINATION

Thursday 17^{th} June 2010, 6pm – 8.10pm
Room 31, Old Main Building

Answer all 4 questions. All are of equal weight.
Small hand-held calculators may be used if they do not incorporate an alphanumeric keyboard.
Answers must be written in ink. Pencils may only be used for drawing, sketching, or graphical work.
One A4 page of notes may be taken into this exam. This must be handwritten. Both sides of the page can be used.
This exam is worth 70% of the total course marks.
This paper may be retained by the candidate.

Some useful formulae for Relativity

Special relativity

Proper time:	$(\Delta\tau)^2 = -(\Delta s)^2$
Time dilation:	$\Delta t = \gamma \Delta\tau$, where $\gamma = \frac{1}{\sqrt{1-v^2}}$
Lorentz contraction:	$l = \frac{l_0}{\gamma}$
Velocity composition law:	$w = \frac{\overline{w}+v}{1+\overline{w}v}$

Lorentz transformations

For a frame moving with velocity v in the x direction: $\Lambda^{\bar{\alpha}}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & & 0 \\ -v\gamma & \gamma & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Inverse: $\Lambda^{v}_{\bar{\beta}} \Lambda^{\bar{\beta}}_{\alpha} = \delta^{v}_{\alpha}$

Lorentz 4-vectors

4-velocity:

$$\vec{U} = \frac{d\vec{x}}{d\tau}$$
$$\vec{U} \cdot \vec{U} = -1$$
$$\vec{U} \underset{MCRF}{\longrightarrow} (1, 0, 0, 0)$$
$$\vec{U} \longrightarrow (\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$$

4-momentum:

$$\vec{p} = m\vec{U} \underset{O}{\longrightarrow} (E, p^x, p^y, p^z)$$
$$E_{obs} = -\vec{p}.\vec{U}_{obs}$$

Tensor analysis in special relativity

Metric tensor: $\vec{e_\alpha} \cdot \vec{e_\beta} = \eta_{\alpha\beta}$

$$(\eta) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Inverse: $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^{\gamma}_{\alpha}$

Covector: $B_\alpha = \eta_{\alpha\beta} B^\beta$

Gradient: $\nabla\phi = \tilde{d}\phi \underset{O}{\longrightarrow} \left\{ \phi_{,\alpha} = \frac{\partial\phi}{\partial x^\alpha} \right\}$

$$\frac{d\phi}{d\tau} = \nabla_{\vec{U}}\phi = \phi_{,\alpha} U^\alpha$$

Perfect fluids

4-vector flux: $\vec{N} = n\vec{U}$

Conservation of particles: $N^{\alpha}_{\ ,\alpha} = 0$

Stress-energy tensor: $\left(T^{\alpha\beta}\right) \underset{MCRF}{\longrightarrow} \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$

i.e. $T^{\alpha\beta} = (\rho + p)\, U^{\alpha}U^{\beta} + p\eta^{\alpha\beta}$

Conservation of 4-momentum: $T^{\alpha\beta}_{\ \ ,\beta} = 0$

General relativity

Christoffel symbols: $\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right)$

$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$

$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right)$

Covariant differentiation: $V^{\alpha}_{\ ;\beta} = V^{\alpha}_{\ ,\beta} + \Gamma^{\alpha}_{\mu\beta} V^{\mu}$

Parallel transport of $\vec{V}$ along $\vec{U}$: $U^{\beta} V^{\alpha}_{\ ;\beta} = 0$

Geodesic: $\frac{d}{d\lambda}\left(\frac{dx^{\alpha}}{d\lambda}\right) + \Gamma^{\alpha}_{\mu\beta}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$

or $m\frac{dp_{\beta}}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^{\nu} p^{\alpha}$

Reimann curvature tensor: $R^{\alpha}_{\ \beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$

$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\ \beta\mu\nu}$

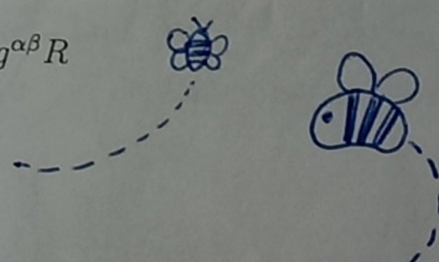
Bianchi identities: $R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$

Ricci tensor: $R_{\alpha\beta} = R^{\mu}_{\ \alpha\mu\beta}$

Ricci scalar: $R = g^{\mu\nu} R_{\mu\nu}$

Einstein tensor: $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$

Einstein field equations: $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$



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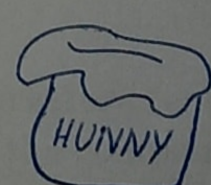
Metric tensors

Interval: $ds^2 = g_{\alpha\beta}\, dx^{\alpha} dx^{\beta}$

Schwarzschild metric: $ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$

Robertson-Walker metric: $ds^2 = -dt^2 + R(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$



Useful constants, units, and formulae:

Gravitational constant	G	=	6.67	×	10^{-11}	$\mathrm{N\ m^2\ kg^{-2}}$
Speed of light	c	=	3.00	×	10^{8}	$\mathrm{m\ s^{-1}}$
Planck constant	h	=	6.63	×	10^{-34}	J s
Boltzmann constant	k	=	1.38	×	10^{-23}	$\mathrm{J\ K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.67	×	10^{-8}	$\mathrm{W\ m^{-2}\ K^{-4}}$
Solar mass	$M_\odot$	=	1.99	×	10^{30}	kg
Solar radius	$R_\odot$	=	6.96	×	10^{8}	m
Earth mass	$M_\oplus$	=	5.98	×	10^{24}	kg
Equatorial radius of Earth	$R_\oplus$	=	6.38	×	10^{6}	m
Mass of moon	M_{moon}	=	7.3	×	10^{22}	kg
Astronomical unit	AU	=	1.50	×	10^{11}	m
Parsec	pc	=	3.09	×	10^{16}	m
Hubble's constant	H_0	=	70			$\mathrm{km\ s^{-1}\ Mpc^{-1}}$

Distance modulus	$m - M$	=	$5 \log d - 5$ (d in pc)
Apparent magnitude	$m_2 - m_1$	=	$2.5 \log \frac{f_1}{f_2}$
For small recession velocities	v/c	=	$\Delta\lambda/\lambda$

Question 1

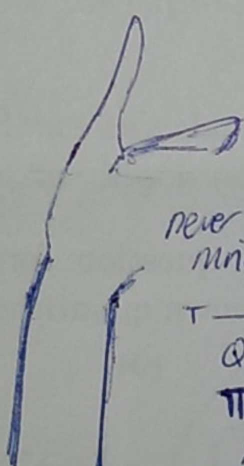
(a) Prove that, in cylindrical co-ordinates (r, θ, z) the line element between two neighbouring points is

$$dl^2 = dr^2 + dz^2 + r^2 d\theta^2$$

(b) The surface of a cone (see diagram) is then specified by the further constraint

$$r = z$$

Use this additional constraint to simplify the line element equation above and hence show that the non-zero components of the metric tensor g on the cone in the co-ordinates (z, θ) are $g_{11} = 2$ and $g_{22} = r^2$.



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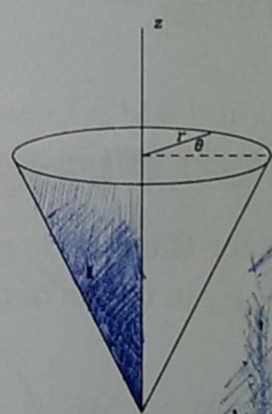


(c) The only non-zero Christoffel symbols are $\Gamma^z_{\theta\theta}$ and $\Gamma^\theta_{\theta z} = \Gamma^\theta_{z\theta}$. Compute the values of these symbols.

(d) In a two-dimensional space, the symmetries of the curvature tensor imply that there is only one independent component. Find the component $R^z_{\theta z\theta}$.

(e) How do you interpret your answer to part (e)?

lol.

Question 2

(a) Explain, in general terms, how the geodesic equation can be used to find the conserved components of a freely falling particle's four momentum, p_α.

(b) The Schwarzschild metric around a neutral non-rotating black hole of mass M is

$$ds^2 = -\left[1 - \frac{2M}{r}\right] dt^2 + \left[1 - \frac{2M}{r}\right]^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Write down the non-zero components of the metric tensor.

(c) Find the non-zero components of $g_{\nu\alpha,\beta}$.

(d) Find the conserved components of p_α.

(e) Use the equation

$$\vec{p}\,.\,\vec{p} = -m^2 = p^\alpha p_\alpha$$

to derive an equation of motion, for the metric given in this question, for a freely falling particle, i.e. find $\left(\frac{dr}{d\tau}\right)^2$, in terms of the other known quantities.

Question 3

(a) Outline the simple pre-relativistic considerations which suggested that black holes might exist. Explain what the *event horizon* is.

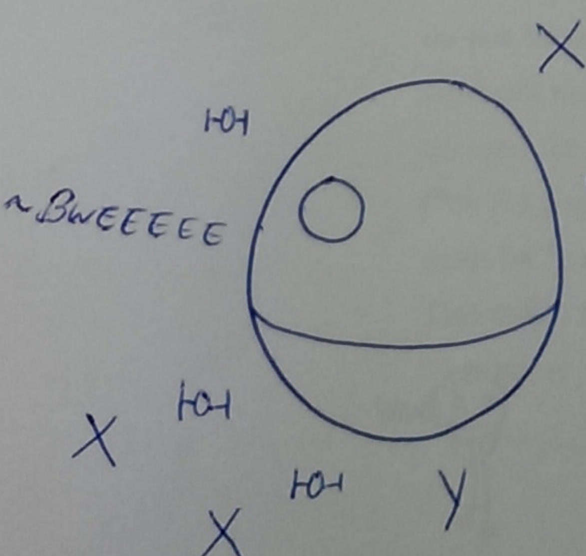
(b) The Stefan-Boltzman law expresses the rate of energy loss per unit area $= \sigma T^4$ from the surface of a black hole. Hawking's expression for the temperature of a black hole is

$$T = \frac{\hbar c^3}{8\pi k G M}$$

Using the two equations above, show how the rate of loss of mass from a black hole depends on its mass.

(c) Use the expression you derived above to obtain the lifetime of a black hole.

(d) Hence calculate the lifetime of a black hole which has a mass equal to that of the Moon (see data sheet in this exam paper).



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Question 4

This is "discuss"-type question, but equations are not forbidden if you wish to make a point in your answer clearer.

(a) What is the weak equivalance principle?

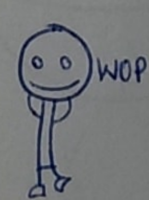
(b) What do you think motivated the introduction of the weak equivalance principle?

(c) What is the strong equivalence principle?

(d) Do Einstein's field equations accommodate the strong equivalence principle? If so, how?

--- x delicious the conqueror.

BEELIEVE IT !



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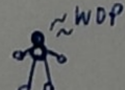
one and only ~.



Lalalalalalalalala.