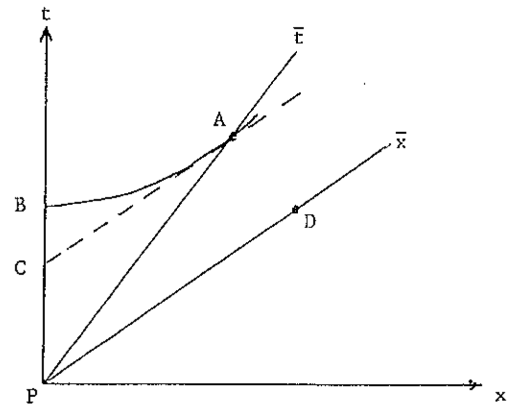
Question 1



In the spacetime diagram shown, t and x are the coordinates as measured by an observer O, with event P as origin. $\bar{t}$ and $\bar{x}$ are the axes corresponding to an observer $\overline{O}$, moving at a speed v relative to O. The coordinates of A are $t, x = \left(\sqrt{2}, 1\right)$ and of D are $\left(1, \sqrt{2}\right)$. The curve AB is an invariant hyperbola and the line AC is a straight line tangent to the hyperbola at A.

(a) What are the coordinates of point B?

(b) What is the elapsed time between events P and A according to observer O?

(c) What is the elapsed proper time between events P and A?

(d) If PA is the world line of the observer $\overline{O}$, what is his velocity relative to O?

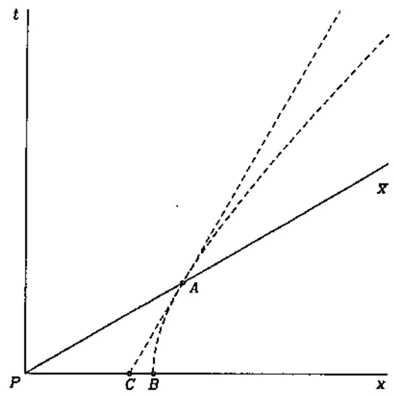
(e) The angles between the line PD and the x axis, and the line PA and the t axis, are equal. What does the line PD correspond to in the frame of observer $\overline{O}$?

(f) What are the coordinates of point C?

Question 1

In the space-time diagram shown, t and x are co-ordinates as measured by observer O, with event P as origin. The co-ordinates of point A are $(1/2, 1)$.

The curve AB is an invariant hyperbola, and AC is a straight line tangent to the hyperbola at A.



(a) Determine the co-ordinates of the event B.

(b) Write down the equation in x and t describing the curve AB. Differentiate this equation to determine the gradient of this curve, and hence determine the slope of the line AC.

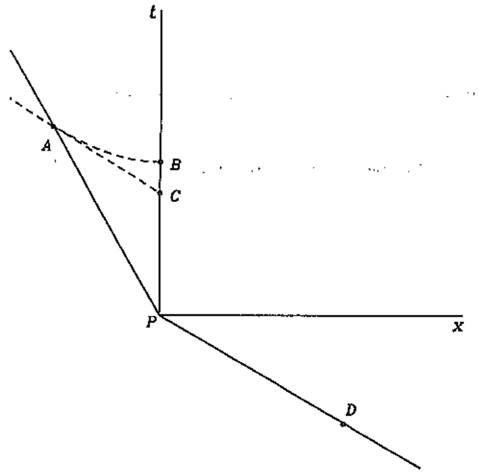
(c) Determine the co-ordinates of the event C.

(d) If PA is the $\bar{x}$ axis for an observer $\bar{O}$, determine his velocity relative to O.

(e) If the events P and A correspond to the two ends of the rod in $\bar{O}$ at time $\bar{t} = 0$, determine the length of the rod as seen by observers $\bar{O}$ and O. Hence show that the rod exhibits the expected Lorentz contraction in frame O.

Question 1

In the space-time diagram shown, t and x are co-ordinates as measured by an observer $\mathcal{O}$, with event P as origin. The co-ordinates of event A are $(t, x) = (\sqrt{3}, -1)$m and of D are $(-1, \sqrt{3})$m. The curve AB is an invariant hyperbola at A. The line AC is a tangent at point A.



(a) What are the co-ordinates of point B?

(b) What is the elapsed time between events P and A according to observer $\mathcal{O}$?

(c) What is the elapsed proper time between events P and A?

(d) If PA is the world line of an observer $\overline{\mathcal{O}}$, what is his 3-velocity relative to $\mathcal{O}$?

(e) What does the line PD correspond to?

(f) What are the co-ordinates of the point C?

Question 1

(a) State the equivalence principle (in both weak and strong forms). Explain what you understand by a freely falling frame.

(b) The metric tensor for a reference frame undergoing constant acceleration, a, can be expressed as (taking $c = 1$)

$$g_{\mu\nu} = \begin{pmatrix} -(1+ax)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that the only non-zero values of the Christoffel symbols for this metric are

$$\Gamma^1_{00} = a(1+ax)$$

and

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{a}{(1+ax)}$$

(c) Using the results of part (b) above, show that the geodesic equations are

$$\ddot{x} + a(1+ax)\dot{t}^2 = 0$$

and

$$(1+ax)\ddot{t} + 2a\dot{t}\dot{x} = 0$$

(d) Briefly comment on what happens if you take the limit of $a \to 0$ in the geodesic equations above. How might this be interpreted?

(e) Demonstrate that for the metric given, the Riemann tensor $R^{\alpha}_{\mu\nu\sigma}$ has only one independent component, and thus reduces to the scalar R.

Question 2

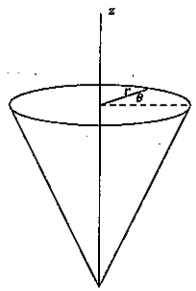
(a) Prove that, in cylindrical co-ordinates (r, θ, z) the line element between two neighbouring points is

$$dl^2 = dr^2 + dz^2 + r^2 d\theta^2$$

(b) The surface of a cone (see diagram) is then specified by the further constraint

$$r = z$$

Use this additional constraint to eliminate r from the line element equation above.



(c) Hence show that the non-zero components of the metric tensor g on the cone in the co-ordinates (z, θ) are $g_{11} = 2$ and $g_{22} = r^2$.

(d) The only non-zero Christoffel symbols are $\Gamma^z_{\theta\theta}$ and $\Gamma^\theta_{\theta z} = \Gamma^\theta_{z\theta}$. Compute the values of these symbols.

(e) In a two-dimensional space, the symmetries of the curvature tensor imply that there is only one independent component. Find the component $R^z_{\theta z\theta}$. How do you interpret your result?

Question 2

(a) Show that the metric for the surface of a sphere of fixed radius r, expressed in polar co-ordinates, has non-zero components:

$$g_{\alpha\beta} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

(b) Calculate all non-zero values of the Christoffel symbols $\Gamma^\mu_{\alpha\beta}$ for the metric in part (a).

(c) Hence calculate the Riemann curvature tensor component $R_{\theta\phi\theta\phi}$ for the metric in part (a).

Question 2

A spatially uniform cloud of dust particles is set in motion by a "solar wind" of photons, so that it moves in the x direction with a velocity

$$v = \frac{\alpha t}{\sqrt{1+\alpha^2 t^2}},$$

where α is a constant, according to an inertial observer $\mathcal{O}$. The number density of particles in the cloud at time $t = 0$ is n_0, as measured in frame $\mathcal{O}$. Each dust particle has rest mass m.

(a) Explain what is meant by the term "dust". Find the components of the 4-velocity $\vec{U}$ of the dust particles in frame $\mathcal{O}$ as functions of time.

(b) Write down the components of the number flux 4-vector $\vec{N}$ of the cloud in frame $\mathcal{O}$, as functions of t and the number density n in the MCRF of the cloud.

(c) Use the law of conservation of particles

$$N^{\alpha}_{\ ,\alpha} = 0$$

to determine the number density n explicitly as a function of n_0 and t.

(d) Compute the non-zero components of the stress-energy tensor $T^{\alpha\beta}$ for the dust cloud, as functions of t.

(e) By examining the components of the energy-momentum conservation law

$$T^{\alpha\beta}_{\ \ ,\beta} = 0$$

find out whether energy and momentum are conserved for this gas cloud. Discuss your findings.

Question 3

Calculate the Riemann curvature tensor of the surface of a cylinder, as follows:

(a) Derive an expression for the line element dl^2 in cylindrical co-ordinates (r, θ, z).

(b) Taking the radius r as fixed, deduce the components of the metric tensor $g_{\alpha\beta}$ in co-ordinates (θ, z) on the surface of a cylinder.

(c) Find the Christoffel symbols in this co-ordinate system.

(d) Hence calculate the Riemann curvature tensor of the surface. Is the $R_{\theta z \theta z}$ you find, what you expect? Explain your answer.

Question 3

(a) Outline the simple pre-relativistic considerations which lead to the expectation that black holes may exist. Explain what the *event horizon* is.

(b) Derive an approximate relationship relating the temperature of Hawking radiation from a black hole, to the black hole mass. Show all working and comment on any assumptions or approximations made.

(c) Use your answer to part (b) above to calculate the temperature in degrees K for a black hole which has the mass of the Earth (5.98×10^{24} kg).

(d) Use the Stefan-Boltzman law (rate of energy loss per unit area = σT^4) to derive an expression for the total rate of loss of energy, dM/dt, from the event horizon of a black hole.

(e) Re-arrange the expression you derived in part (d) above, and integrate it to obtain an expression for the lifetime of a black hole. Thus calculate the lifetime of a black hole which has a mass equal to that of the Earth. Comment on the importance of your answer given the current estimate for the age of the Universe of $13 - 14 \times 10^9$ years.

Question 3

(a) Calculate the Schwarzschild radius r_0 of a neutral non-rotating black hole of mass 4×10^7 solar masses. (Take $G = 6.67 \times 10^{-8}$ gm^{-1}cm^{3}s^{-2} and $c = 3.0 \times 10^{10}$cm s^{-1}and one solar mass as 2×10^{33}gm).

(b) For an object in orbit at a radius r around the neutral non-rotating black hole, prove that

$$r = r_j \left[1 \pm \left(1 - \frac{3r_0}{r_j} \right)^{1/2} \right]$$

where r_0 is the Schwarzschild radius, $r_j = J^2/c^2 r_0$ and J is the angular momentum per unit mass. Hence calculate the numerical value of the smallest possible stable orbit r_{min}.

(c) Derive an expression for the tangential velocity of the object orbiting at r_{min}. Hence calculate the period of the orbit, as measured by a local observer at rest.

(d) If orbiting matter is falling onto the neutral non-rotating black hole, derive an expression for the fraction of the rest mass converted into energy.

(e) Assume that black hole is to deliver a power output of 2×10^{40} Watts, and the total amount of mass available for consumption is equal to 10 percent of the mass of the galaxy in which the black hole resides. Calculate the expected lifetime of the power output. [Take the mass of the galaxy to be 10^{10} solar masses, and one solar mass as 2×10^{33} gm].

Question 3

(a) Calcualte the metric g_{ij} for the surface of a sphere of fixed radius r using spherical polar angles (θ, ϕ) as co-ordinates.

(b) Calculate the Riemann curvature tensor $R_{\theta\phi\theta\phi}$ of the sphere and use the symmetry properties of the tensor to show for this case that all other components can be deduced from this component.

Question 3

The Robertson-Walker line element is given by

$$ds^2 = c^2dt^2 - R(t)^2\left[\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2 d\Omega^2\right]$$

(a) Write down the meaning of the all the symbols (other than c) on the right hand side of the equation above.

(b) What is the Cosmological Principle?

(c) What is the empirical justification is for the Cosmological Principle?

(d) Is the Cosmological Principle incorporated into the Robertson-Walker metric? If so, explain how.

(e) Outline in words only (no detailed mathematics is required) how the Robertson-Walker metric together with the the field equations can be used to understand the expansion history of the Universe (i.e. how the Friedmann equation is derived).

(f) Provide a sketch illustrating the expansion history of the universe.

Question 4

The Robertson-Walker line element is given by

$$ds^2 = c^2dt^2 - R(t)^2 \left[\frac{d\sigma^2}{1-k\sigma^2} + \sigma^2 d\Omega^2\right]$$

(a) Write down the meaning of the symbols in the equation above.

(b) Discuss any observation or observations which give an empirical justification for the Cosmological Principle.

(c) Comment on whether or not the Cosmological Principle is incorporated into the Robertson-Walker metric.

(d) The constant k can take values $1, 0, -1$. Explain what these values correspond to in terms of models of the Universe.

(e) Outline in words (no mathematics is required) how the Robertson-Walker metric together with the the Einstein field equations can be used to understand the expansion history of the Universe. Provide a sketch illustrating how the scale-factor of the Universe has evolved with time.

Question 4

The Schwarzschild metric around a neutral non-rotating black hole of mass M is

$$ds^2 = -\left[1-\frac{2M}{r}\right]dt^2 + \left[1-\frac{2M}{r}\right]^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

(a) Write down the non-zero components of the metric tensor.

(b) If a particle of rest mass m travels in the equatorial plane ($\theta = \pi/2$) with conserved momentum components

$$p_t = -mE,\ p_\phi = mL,\ p_\theta = 0,$$

use the equation

$$\vec{p}.\vec{p} = -m^2$$

to derive an equation of motion for the particle.

(c) Discuss the significance of the surface $r = 2M$ in this metric.

Question 4

(a) Outline the simple pre-relativistic considerations which lead to the expectation that black holes may exist. Explain what the *event horizon* is.

(b) Derive an expression for the *proper time*, τ, as a function of radial distance, r, from the singularity at the centre of a non-rotating neutral black hole, as measured by an observer on a spaceship approaching the black hole.

(c) Provide a sketch of the function above (plotting r on the y axis and τ on the x axis). On the same diagram, sketch the form of $r(t)$, where t is the time measured by a *remote* observer.

(d) Comment on what happens to the observer on the spaceship as he crosses the event horizon.

(e) The one dimensional energy equation for a particle in motion about a non-rotating neutral black hole is

$$\frac{E^2}{m^2c^4} = \frac{(dr/d\tau)^2}{c^2} + Z\left(1 + \frac{J^2}{c^2r^2}\right)$$

Comment on the significance of the last term in the equation above. Provide a sketch of this term as a function of radial distance and discuss its interpretation.

(f) Calculate the energy released, as a fraction of the rest mass, when a particle coming from infinity moves into the lowest stable circular orbit around the black hole.

Question 4

(a) Outline the simple pre-relativistic considerations which lead to the expectation that black holes may exist. Explain what the *event horizon* is.

(b) The Stefan-Boltzman law expresses the rate of energy loss per unit area $= \sigma T^4$ from the surface of a black hole. Hawking's expression for the temperature of a black hole is

$$T = \frac{\hbar c^3}{8\pi kGM}$$

Use the two equations above to show that the total rate of loss of mass from the event horizon of a black hole is inversely proprtional to the square of the mass of the black hole.

(c) Use the full expression you derived in part (b) above to obtain the lifetime of a black hole.

(d) Hence calculate the lifetime of a black hole which has a mass equal to that of the Earth (see data sheet in this exam paper). Briefly comment on your answer, given that the age of the Universe today is $\sim 14 \times 10^9$ years.

Question 3

The Schwarzchild metric, expressed in Eddington-Finkelstein co-ordinates, is

$$ds^2 = \left(1 - \frac{r_o}{r}\right) c^2 d\tilde{t}^{\,2} - 2cdr d\tilde{t}\left(\frac{r_o}{r}\right) - dr^2\left(1 + \frac{r_o}{r}\right) + r^2 d\Omega^2$$

(a) Explain why the Eddington-Finkelstein co-ordinate system is useful.

(b) Use the metric above to obtain an equation describing a radial light ray.

(c) Verify that your answer in part (b) satisfies the solutions

$$\frac{d\tilde{t}}{dr} = -\frac{1}{c}$$

$$\frac{d\tilde{t}}{dr} = \frac{1}{c}\left(\frac{1 + r_o/r}{1 - r_o/r}\right)$$

where r, r_o are the radial distance and Schwarzchild radius respectively.

(d) Draw a space-time diagram illustrating the solutions in part (c). Give a brief physical interpretation for this diagram.

(e) An astronaut is using the engines of a spacecraft to remain *at rest* at radius R away from a star (considered as a point source of mass M). Although the spacecraft is at co-ordinate rest, it is accelerating with respect to any local inertial frame. Derive an expression for the invariant acceleration.

[**HINTS for part (e):** Use Schwarzchild co-ordinates - it's easier. The invariant acceleration of a particle in a curved space-time is found by taking the covariant derivative of the four velocity (which therefore involves the Christoffel symbols, which can be expressed in terms of the metric components). The four-velocity can be obtained from the line element.]

Question 2

(a) The cosmic microwave background radiation (CMB) has been shown to have a black body spectrum. We can assume that the CMB "photon gas" has a random distribution of velocities at any point, ie. there is no prefered direction in the MCRF.

Give the components of the stress-energy tensor for the CMB and hence show that $p = \rho/3$, where p is pressure and ρ is energy density.

(b) A spatially uniform cloud of dust particles is being accelerated by a "solar wind" of photons, so that, according to an inertial observer O, it moves in the x direction with a velocity

$$v = \frac{\alpha t}{\sqrt{1 + \alpha^2 t^2}}$$

where α is a constant. The number density of particles in the cloud at time $t = 0$ is n_o, as measured in frame O. Each dust particle has rest mass m.

(i) Find the components of the 4-velocity $\vec{U}$ of the dust particles in frame O, as functions of time.

(ii) Write down the components of the number flux 4-vector $\vec{N}$ of the cloud in frame O, as functions of t and the number density n in the MCRF of the cloud.

(iii) Compute the non-zero components of the stress-energy tensor $T^{\alpha\beta}$ for the dust cloud, as functions of t.