UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

MID-SESSION TEST

Wednesday 20^{th} April 2016

PHYS3550 General Relativity

Time allowed – 50 minutes.

Total number of questions – 2.

Answer both questions.

Both questions are of equal value.

One A4 page of notes, handwritten on one side-only, may be taken into this test.

Candidates must supply their own, university approved, calculator.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Candidates may keep this paper.

Some useful formulae for Relativity

Special relativity

Proper time:	$(\Delta\tau)^2 = -(\Delta s)^2$
Time dilation:	$\Delta t = \gamma\Delta\tau$, where $\gamma = \frac{1}{\sqrt{1-v^2}}$
Lorentz contraction:	$l = \frac{l_0}{\gamma}$
Velocity composition law:	$w = \frac{\overline{w}+v}{1+\overline{w}v}$

Lorentz transformations

For a frame moving with velocity v in the x direction: $\Lambda^{\overline{\alpha}}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & & 0 \\ -v\gamma & \gamma & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Inverse: $\Lambda^{v}_{\overline{\beta}}\Lambda^{\overline{\beta}}_{\alpha} = \delta^{v}_{\alpha}$

Lorentz 4-vectors

4-velocity:

$$\overrightarrow{U} = \frac{d\overrightarrow{x}}{d\tau}$$
$$\overrightarrow{U}\cdot\overrightarrow{U} = -1$$
$$\overrightarrow{U} \underset{MCRF}{\longrightarrow} (1,0,0,0)$$
$$\overrightarrow{U} \longrightarrow (\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$$

4-momentum:

$$\overrightarrow{p} = m\overrightarrow{U} \underset{O}{\longrightarrow} (E, p^x, p^y, p^z)$$
$$E_{obs} = -\overrightarrow{p}.\overrightarrow{U}_{obs}$$

Tensor analysis in special relativity

Metric tensor:

$$\overrightarrow{e_\alpha}\cdot\overrightarrow{e_\beta} = \eta_{\alpha\beta}$$
$$(\eta) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Inverse: $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^{\gamma}_{\alpha}$

Covector: $B_\alpha = \eta_{\alpha\beta}B^\beta$

Gradient:

$$\nabla\phi = \tilde{d}\phi \underset{O}{\longrightarrow} \left\{\phi_{,\alpha} = \frac{\partial\phi}{\partial x^\alpha}\right\}$$
$$\frac{d\phi}{d\tau} = \nabla_{\overrightarrow{U}}\phi = \phi_{,\alpha}U^\alpha$$

Perfect fluids

4-vector flux:	$\vec{N} = n\vec{U}$
Conservation of particles:	$N^{\alpha}_{,\alpha} = 0$
Stress-energy tensor:	$\left(T^{\alpha\beta}\right) \underset{MCRF}{\longrightarrow} \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$
	i.e. $T^{\alpha\beta} = (\rho + p)\, U^{\alpha} U^{\beta} + p\eta^{\alpha\beta}$
Conservation of 4-momentum:	$T^{\alpha\beta}_{,\beta} = 0$

General relativity

Christoffel symbols:	$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right)$
	$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$
	$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right)$
Covariant differentiation:	$V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta} V^{\mu}$
Parallel transport of $\vec{V}$ along $\vec{U}$:	$U^{\beta} V^{\alpha}_{;\beta} = 0$
Geodesic:	$\frac{d}{d\lambda}\left(\frac{dx^{\alpha}}{d\lambda}\right) + \Gamma^{\alpha}_{\mu\beta} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\beta}}{d\lambda} = 0$
Reimann curvature tensor:	$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$
	$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\mu\nu}$
Bianchi identities:	$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$
Ricci tensor:	$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$
Ricci scalar:	$R = g^{\mu\nu} R_{\mu\nu}$
Einstein tensor:	$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$
Einstein field equations:	$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$
Schwarzschild radius:	$r_s = 2GM/c^2$

Metric tensors

Interval:	$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$
Schwarzschild metric:	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$
Robertson-Walker metric:	$ds^2 = -dt^2 + R(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$

Other things

Uncertainty Principle: $\Delta p \approx \hbar / r$

Useful constants, units, and formulae:

Gravitational constant	G	=	6.67	×	10^{-11}	N m^2 kg^{-2}
Speed of light	c	=	3.00	×	10^{8}	m s^{-1}
Planck constant	h	=	6.63	×	10^{-34}	J s
Boltzmann constant	k	=	1.38	×	10^{-23}	J K^{-1}
Stefan-Boltzmann constant	σ	=	5.67	×	10^{-8}	W m^{-2} K^{-4}
Solar mass	$M_\odot$	=	1.99	×	10^{30}	kg
Solar radius	$R_\odot$	=	6.96	×	10^{8}	m
Earth mass	$M_\oplus$	=	5.98	×	10^{24}	kg
Equatorial radius of Earth	$R_\oplus$	=	6.38	×	10^{6}	m
Mass of moon	M_{moon}	=	7.3	×	10^{22}	kg
Astronomical unit	AU	=	1.50	×	10^{11}	m
Parsec	pc	=	3.09	×	10^{16}	m
Hubble's constant	H_0	=	70			km s^{-1} Mpc^{-1}

Distance modulus	$m - M$	=	$5 \log d - 5$	(d in pc)
Apparent magnitude	$m_2 - m_1$	=	$2.5 \log \frac{f_1}{f_2}$	
For small recession velocities	v/c	=	$\Delta\lambda/\lambda$	

Question 1

Calculate the Riemann curvature tensor of the surface of a cylinder, as follows:

(a) Derive an expression for the line element dl^2 in cylindrical co-ordinates (r, θ, z).

(b) Taking the radius r as fixed, deduce the components of the metric tensor $g_{\alpha\beta}$ in co-ordinates (θ, z) on the surface of a cylinder.

(c) Find the Christoffel symbols in this co-ordinate system.

(d) Hence calculate the Riemann curvature tensor of the surface. Is the $R_{\theta z \theta z}$ you find, what you expect? Explain your answer.

Question 2

The total number of components in the Riemann curvature tensor $R_{\alpha\beta\mu\nu}$ is $4 \times 4 \times 4 \times 4 = 256$.

The following identities can be derived using the metric components (you are not asked to derive them):

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$$

Use these identities to show that the number of *independent* components in $R_{\alpha\beta\mu\nu}$ is actually only 21.

Explain each step of your reasoning clearly and carefully.

(Hint: treat pairs of indices. Calculate how many independent choices of pairs there are for the first and the second pairs on $R_{\alpha\beta\mu\nu}$.)