UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

Session 1 2013

PHYS3550 General Relativity

Time allowed – 2 hours.

Total number of questions – 4.

Answer all questions.

All questions are of equal value.

This exam is worth 70% of the final grade.

One A4 page of notes, handwritten on one side-only, may be taken into this exam.
Candidates must supply their own, university approved, calculator.
Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Candidates may keep this paper.

Some useful formulae for Relativity

Special relativity

Proper time:	$(\Delta\tau)^2 = -(\Delta s)^2$
Time dilation:	$\Delta t = \gamma \Delta\tau$, where $\gamma = \frac{1}{\sqrt{1-v^2}}$
Lorentz contraction:	$l = \frac{l_0}{\gamma}$
Velocity composition law:	$w = \frac{\overline{w}+v}{1+\overline{w}v}$

Lorentz transformations

For a frame moving with velocity v in the x direction: $\Lambda^{\overline{\alpha}}_{\ \beta} = \begin{pmatrix} \gamma & -v\gamma & & 0 \\ -v\gamma & \gamma & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Inverse: $\Lambda^{\underline{v}}_{\ \overline{\beta}} \Lambda^{\overline{\beta}}_{\ \alpha} = \delta^{v}_{\ \alpha}$

Lorentz 4-vectors

4-velocity:

$\overrightarrow{U} = \frac{d\overrightarrow{x}}{d\tau}$

$\overrightarrow{U} \cdot \overrightarrow{U} = -1$

$\overrightarrow{U} \underset{MCRF}{\longrightarrow} (1,0,0,0)$

$\overrightarrow{U} \longrightarrow (\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$

4-momentum:

$\overrightarrow{p} = m\overrightarrow{U} \underset{O}{\longrightarrow} (E, p^x, p^y, p^z)$

$E_{obs} = -\overrightarrow{p} . \overrightarrow{U}_{obs}$

Tensor analysis in special relativity

Metric tensor: $\overrightarrow{e_\alpha} \cdot \overrightarrow{e_\beta} = \eta_{\alpha\beta}$

$$(\eta) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Inverse: $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^{\gamma}_{\ \alpha}$

Covector: $B_\alpha = \eta_{\alpha\beta} B^\beta$

Gradient: $\nabla\phi = \tilde{d}\phi \underset{O}{\longrightarrow} \left\{\phi_{,\alpha} = \frac{\partial\phi}{\partial x^\alpha}\right\}$

$\frac{d\phi}{d\tau} = \nabla_{\overrightarrow{U}}\phi = \phi_{,\alpha} U^\alpha$

Perfect fluids

4-vector flux: $\vec{N} = n\vec{U}$

Conservation of particles: $N^{\alpha}_{,\alpha} = 0$

Stress-energy tensor: $\left(T^{\alpha\beta}\right) \underset{MCRF}{\longrightarrow} \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$

i.e. $T^{\alpha\beta} = (\rho + p)\, U^{\alpha}U^{\beta} + p\eta^{\alpha\beta}$

Conservation of 4-momentum: $T^{\alpha\beta}_{,\beta} = 0$

General relativity

Christoffel symbols: $\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right)$

$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$

$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right)$

Covariant differentiation: $V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta}V^{\mu}$

Parallel transport of $\vec{V}$ along $\vec{U}$: $U^{\beta}V^{\alpha}_{;\beta} = 0$

Geodesic: $\frac{d}{d\lambda}\left(\frac{dx^{\alpha}}{d\lambda}\right) + \Gamma^{\alpha}_{\mu\beta}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$

Reimann curvature tensor: $R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$

$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda}R^{\lambda}_{\beta\mu\nu}$

Bianchi identities: $R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$

Ricci tensor: $R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$

Ricci scalar: $R = g^{\mu\nu}R_{\mu\nu}$

Einstein tensor: $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$

Einstein field equations: $G^{\alpha\beta} = 8\pi T^{\alpha\beta}$

Schwarzschild radius: $r_s = 2GM/c^2$

Metric tensors

Interval: $ds^2 = g_{\alpha\beta}dx^{\alpha}dx^{\beta}$

Schwarzschild metric: $ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$

Robertson-Walker metric: $ds^2 = -dt^2 + R(t)^2\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$

Other things

Uncertainty Principle: $\Delta p \approx \hbar/r$

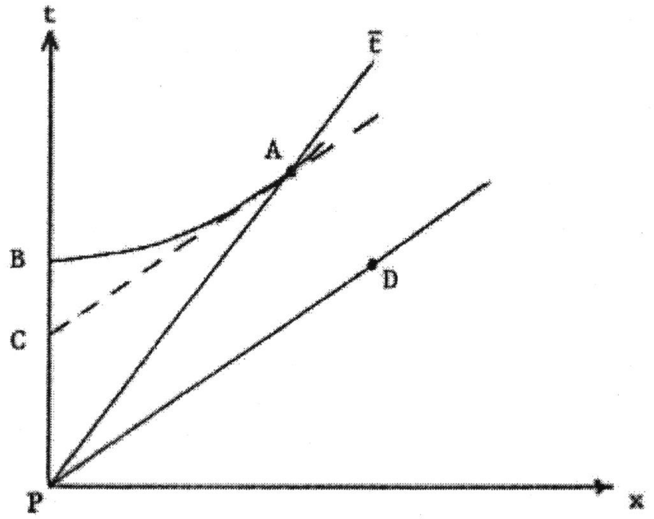
Useful constants, units, and formulae:

Gravitational constant	G	=	6.67	×	10^{-11}	$\mathrm{N\ m^2\ kg^{-2}}$
Speed of light	c	=	3.00	×	10^{8}	$\mathrm{m\ s^{-1}}$
Planck constant	h	=	6.63	×	10^{-34}	J s
Boltzmann constant	k	=	1.38	×	10^{-23}	$\mathrm{J\ K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.67	×	10^{-8}	$\mathrm{W\ m^{-2}\ K^{-4}}$
Solar mass	$M_\odot$	=	1.99	×	10^{30}	kg
Solar radius	$R_\odot$	=	6.96	×	10^{8}	m
Earth mass	$M_\oplus$	=	5.98	×	10^{24}	kg
Equatorial radius of Earth	$R_\oplus$	=	6.38	×	10^{6}	m
Mass of moon	M_{moon}	=	7.3	×	10^{22}	kg
Astronomical unit	AU	=	1.50	×	10^{11}	m
Parsec	pc	=	3.09	×	10^{16}	m
Hubble's constant	H_0	=	70			$\mathrm{km\ s^{-1}\ Mpc^{-1}}$

Distance modulus	$m - M$	=	$5 \log d - 5$ (d in pc)
Apparent magnitude	$m_2 - m_1$	=	$2.5 \log \frac{f_1}{f_2}$
For small recession velocities	v/c	=	$\Delta\lambda/\lambda$

Question 1

In the spacetime diagram shown, t and x are the coordinates as measured by an observer O, with event P as origin. $\bar{t}$ and $\bar{x}$ are the axes corresponding to an observer $\overline{O}$, moving at a speed v relative to O. The coordinates of A are $(t, x) = \left(\sqrt{2}, 1\right)$ and of D are $\left(1, \sqrt{2}\right)$. The curve AB is an invariant hyperbola and the line AC is a straight line tangent to the hyperbola at A.



(a) What are the coordinates of point B?

(b) What is the elapsed time between events P and A according to observer O?

(c) What is the elapsed proper time between events P and A?

(d) If PA is the world line of the observer $\overline{O}$, what is his velocity relative to O?

(e) The angles between the line PD and the x axis, and the line PA and the t axis, are equal. What does the line PD correspond to in the frame of observer $\overline{O}$?

(f) What are the coordinates of point C?

Question 2

(a) Show that the metric for the surface of a sphere of fixed radius r, expressed in spherical polar co-ordinates, has non-zero components:

$$g_{\alpha\beta} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2\theta \end{pmatrix}$$

(b) Calculate all non-zero values of the Christoffel symbols $\Gamma^{\mu}_{\alpha\beta}$ for the metric in part (a).

(c) Hence calculate the Riemann curvature tensor component $R_{\theta\phi\theta\phi}$ for the metric in part (a).

Question 3

The Friedmann-Robertson-Walker line element is given by

$$ds^2 = c^2 dt^2 - R(t)^2 \left[\frac{d\sigma^2}{1 - k\sigma^2} + \sigma^2 d\Omega^2 \right]$$

(a) Write down the meaning of all the symbols (other than c) on the right hand side of the equation above.

(b) What is the Cosmological Principle?

(c) What is the empirical justification for the Cosmological Principle?

(d) Is the Cosmological Principle incorporated into the Friedman-Robertson-Walker metric? If so, explain how.

(e) Outline in words only (no detailed mathematics is required) how the Friedman-Robertson-Walker metric together with the the field equations can be used to understand the expansion history of the Universe (i.e. how the Friedmann equation is derived).

(f) Provide a sketch illustrating the expansion history of the universe.

Question 4

(a) Outline the simple pre-relativistic considerations which lead to the expectation that black holes may exist. Explain what the *event horizon* is.

(b) The Stefan-Boltzman law expresses the rate of energy loss per unit area $= \sigma T^4$ from the surface of a black hole. Hawking's expression for the temperature of a black hole is

$$T = \frac{\hbar c^3}{8\pi k G M}$$

Use the two equations above to show that the total rate of loss of mass from the event horizon of a black hole is inversely proprtional to the square of the mass of the black hole.

(c) Use the full expression you derived in part (b) above to obtain the lifetime of a black hole.

(d) Hence calculate the lifetime of a black hole which has a mass equal to that of the Earth (see data sheet in this exam paper). Briefly comment on your answer, given that the age of the Universe today is $\sim 14 \times 10^9$ years.