UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

JUNE 2012

PHYS3550 General Relativity

Time allowed – 2 hours.

Total number of questions – 4.

Answer all questions.

All questions are of equal value.

This exam is worth 70% of the final grade.

One A4 page of notes, handwritten on one side-only, may be taken into this exam.
Candidates must supply their own, university approved, calculator.
Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work.

Candidates may keep this paper.

Some useful formulae for Relativity

Special relativity

Proper time:	$(\Delta\tau)^2 = -(\Delta s)^2$
Time dilation:	$\Delta t = \gamma\Delta\tau$, where $\gamma = \frac{1}{\sqrt{1-v^2}}$
Lorentz contraction:	$l = \frac{l_0}{\gamma}$
Velocity composition law:	$w = \frac{\overline{w}+v}{1+\overline{w}v}$

Lorentz transformations

For a frame moving with velocity v in the x direction: $\Lambda^{\bar{\alpha}}_{\beta} = \begin{pmatrix} \gamma & -v\gamma & & 0 \\ -v\gamma & \gamma & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

Inverse: $\Lambda^{\nu}_{\bar{\beta}}\Lambda^{\bar{\beta}}_{\alpha} = \delta^{\nu}_{\alpha}$

Lorentz 4-vectors

4-velocity:

$$\vec{U} = \frac{d\vec{x}}{d\tau}$$
$$\vec{U}\cdot\vec{U} = -1$$
$$\vec{U} \underset{MCRF}{\longrightarrow} (1,0,0,0)$$
$$\vec{U} \longrightarrow (\gamma, \gamma v^x, \gamma v^y, \gamma v^z)$$

4-momentum:

$$\vec{p} = m\vec{U} \underset{O}{\longrightarrow} (E, p^x, p^y, p^z)$$
$$E_{obs} = -\vec{p}\,.\,\vec{U}_{obs}$$

Tensor analysis in special relativity

Metric tensor: $\vec{e_\alpha}\cdot\vec{e_\beta} = \eta_{\alpha\beta}$

$$(\eta) = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Inverse: $\eta_{\alpha\beta}\eta^{\beta\gamma} = \delta^{\gamma}_{\alpha}$

Covector: $B_\alpha = \eta_{\alpha\beta}B^\beta$

Gradient: $\nabla\phi = \tilde{d}\phi \underset{O}{\longrightarrow} \left\{\phi_{,\alpha} = \frac{\partial\phi}{\partial x^\alpha}\right\}$

$$\frac{d\phi}{d\tau} = \nabla_{\vec{U}}\phi = \phi_{,\alpha}U^\alpha$$

Perfect fluids

4-vector flux:	$\overrightarrow{N} = n\overrightarrow{U}$
Conservation of particles:	$N^{\alpha}_{,\alpha} = 0$
Stress-energy tensor:	$\left(T^{\alpha\beta}\right) \underset{MCRF}{\longrightarrow} \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$
	i.e. $T^{\alpha\beta} = (\rho + p)\, U^{\alpha}U^{\beta} + p\eta^{\alpha\beta}$
Conservation of 4-momentum:	$T^{\alpha\beta}_{,\beta} = 0$

General relativity

Christoffel symbols:	$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right)$
	$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$
	$\Gamma_{\mu\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right)$
Covariant differentiation:	$V^{\alpha}_{;\beta} = V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta} V^{\mu}$
Parallel transport of $\overrightarrow{V}$ along $\overrightarrow{U}$:	$U^{\beta} V^{\alpha}_{;\beta} = 0$
Geodesic:	$\frac{d}{d\lambda}\left(\frac{dx^{\alpha}}{d\lambda}\right) + \Gamma^{\alpha}_{\mu\beta} \frac{dx^{\mu}}{d\lambda}\frac{dx^{\beta}}{d\lambda} = 0$
Reimann curvature tensor:	$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\sigma\mu}\Gamma^{\sigma}_{\beta\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\beta\mu}$
	$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda} R^{\lambda}_{\beta\mu\nu}$
Bianchi identities:	$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0$
Ricci tensor:	$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}$
Ricci scalar:	$R = g^{\mu\nu} R_{\mu\nu}$
Einstein tensor:	$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R$
Einstein field equations:	$G^{\alpha\beta} = 8\pi T^{\alpha\beta}$
Schwarzschild radius:	$r_s = 2GM/c^2$

Metric tensors

Interval:	$ds^2 = g_{\alpha\beta} dx^{\alpha} dx^{\beta}$
Schwarzschild metric:	$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$
Robertson-Walker metric:	$ds^2 = -dt^2 + R(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$

Other things

Uncertainty Principle: $\Delta p \approx \hbar/r$

Useful constants, units, and formulae:

Gravitational constant	G	=	6.67	×	10^{-11}	$\mathrm{N\ m^2\ kg^{-2}}$
Speed of light	c	=	3.00	×	10^{8}	$\mathrm{m\ s^{-1}}$
Planck constant	h	=	6.63	×	10^{-34}	J s
Boltzmann constant	k	=	1.38	×	10^{-23}	$\mathrm{J\ K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.67	×	10^{-8}	$\mathrm{W\ m^{-2}\ K^{-4}}$
Solar mass	$M_\odot$	=	1.99	×	10^{30}	kg
Solar radius	$R_\odot$	=	6.96	×	10^{8}	m
Earth mass	$M_\oplus$	=	5.98	×	10^{24}	kg
Equatorial radius of Earth	$R_\oplus$	=	6.38	×	10^{6}	m
Mass of moon	M_{moon}	=	7.3	×	10^{22}	kg
Astronomical unit	AU	=	1.50	×	10^{11}	m
Parsec	pc	=	3.09	×	10^{16}	m
Hubble's constant	H_0	=	70			$\mathrm{km\ s^{-1}\ Mpc^{-1}}$

Distance modulus	$m - M$	=	$5 \log d - 5$	(d in pc)
Apparent magnitude	$m_2 - m_1$	=	$2.5 \log \frac{f_1}{f_2}$	
For small recession velocities	v/c	=	$\Delta\lambda/\lambda$	

Question 1

A particle of rest mass m and four-momentum $\vec{p}$ moves along in the x direction in some frame. It is observed by an observer $\bar{\mathcal{O}}$ who moves along the x direction with four-velocity $\overrightarrow{\mathcal{U}_{obs}}$.

(a) Write down the four-velocity and the four-momentum of the particle in $\bar{\mathcal{O}}$'s frame and hence show that the energy of the particle as measured by observer $\bar{\mathcal{O}}$ is

$$\bar{E} = -\vec{p}\,.\vec{\mathcal{U}}_{obs}$$

(b) Show that the rest mass observer $\bar{\mathcal{O}}$ attributes to the particle is

$$m = (-\vec{p}\,.\,\vec{p})^{1/2}$$

(c) Show that the ordinary momentum (i.e. the x component of the particle's four-momentum) the observer $\bar{\mathcal{O}}$ measures is

$$\left[\vec{p}\,.\,\vec{p} + \left(\vec{p}\,.\vec{\mathcal{U}}_{obs}\right)^2\right]^{1/2}$$

(d) Show that the ordinary velocity the observer measures is

$$\left[1 + \frac{\vec{p}\,.\,\vec{p}}{\left(\vec{p}\,.\vec{\mathcal{U}}_{obs}\right)^2}\right]^{1/2}$$

Question 2

(a) State the equivalence principle (in both weak and strong forms). Explain what you understand by a freely falling frame.

(b) The metric tensor for a reference frame undergoing constant acceleration, a, can be expressed as (taking $c = 1$)

$$g_{\mu\nu} = \begin{pmatrix} -(1-ax)^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Show that the only non-zero values of the Christoffel symbols for this metric are

$$\Gamma^1_{00} = a(1+ax)$$

and

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{a}{(1+ax)}$$

(c) Using the results of part (b) above, show that the geodesic equations are

$$\ddot{x} + a(1+ax)\dot{t}^2 = 0$$

and

$$(1+ax)\ddot{t} + 2a\dot{t}\dot{x} = 0$$

(d) Briefly comment on what happens if you take the limit of $a \to 0$ in the geodesic equations above. How might this be interpreted?

Question 3

(a) Show that the metric for the surface of a sphere of fixed radius r, expressed in polar co-ordinates, has non-zero components:

$$g_{\alpha\beta} = \begin{pmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{pmatrix}$$

(b) Calculate all non-zero values of the Christoffel symbols $\Gamma^{\mu}_{\alpha\beta}$ for the metric in part (a).

(c) Hence calculate the Riemann curvature tensor component $R_{\theta\phi\theta\phi}$ for the metric in part (a).

(d) Using the symmetry properties of the Riemann tensor, show that for a sphere, all other components can be deduced from from $R_{\theta\phi\theta\phi}$.

Question 4

(a) For an object in orbit at a radius r around the neutral non-rotating black hole, prove that

$$r = r_J \left[1 \pm \left(1 - \frac{3r_0}{r_J} \right)^{1/2} \right]$$

where r_0 is the Schwarzschild radius, $r_J = J^2/c^2 r_0$ and J is the angular momentum per unit mass. Hence show that the value of the smallest possible stable orbit occurs when $r_0 = r_J/3$.

(b) Derive an expression for the tangential velocity of the object orbiting at r_{min}. Hence derive an expression for the period of the orbit, as measured by a local observer at rest.

(c) If orbiting matter is falling onto the neutral non-rotating black hole, derive an expression for the fraction of the rest mass converted into energy.

(d) Assume a black hole is to deliver a power output of 10^{42} Watts and the total amount of mass available for consumption is equal to 10 percent of the mass of the galaxy in which the black hole resides. Calculate the expected lifetime of the power output. [Take the mass of the galaxy to be 10^{11} solar masses and one solar mass as 2×10^{30} kg].