THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF PHYSICS

FINAL EXAMINATION

NOVEMBER 2012

PHYS3510

Advanced Mechanics, Fields and Chaos

Time Allowed – 2 hours

Total number of questions - 4

Answer ALL questions

All questions ARE of equal value

Candidates may bring their own approved calculators.

Answers must be written in ink. Except where they are expressly required, pencils may only be used for drawing, sketching or graphical work

FORMULA SHEET

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk} = Q_k$$

Polar coordinates

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad T = \tfrac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Spherical polar coordinates

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad T = \tfrac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$
$$z = r\cos\theta$$

Canonical Transformations

1) $F_1(q,Q,t)$ $\quad p_i = \frac{\partial F_1}{\partial q_i}$ $\quad P_i = -\frac{\partial F_1}{\partial Q_i}$ $\quad K = H + \frac{\partial F_1}{\partial t}$

2) $F_2(q,P,t)$ $\quad p_i = \frac{\partial F_2}{\partial q_i}$ $\quad Q_i = \frac{\partial F_2}{\partial P_i}$ $\quad K = H + \frac{\partial F_2}{\partial t}$

3) $F_3(p,Q,t)$ $\quad q_i = -\frac{\partial F_3}{\partial p_i}$ $\quad P_i = -\frac{\partial F_3}{\partial Q_i}$ $\quad K = H + \frac{\partial F_3}{\partial t}$

4) $F_4(p,P,t)$ $\quad q_i = -\frac{\partial F_4}{\partial p_i}$ $\quad Q_i = \frac{\partial F_4}{\partial P_i}$ $\quad K = H + \frac{\partial F_4}{\partial t}$

Poisson Bracket $\quad [u,v]_{q,p} = \sum_k \left(\frac{\partial u}{\partial q_k}\frac{\partial v}{\partial p_k} - \frac{\partial u}{\partial p_k}\frac{\partial v}{\partial q_k}\right)$

Hamilton-Jacobi Theory	$H(q,p,t)$	$H(q,p) = \text{constant}$
Canonical transformation to	$Q_i,\ P_i$	P_i (constant of motion)
New Hamiltonian	$K = 0$	$K = H(P_i) = \alpha_1$
New equations of motion	$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0$ $\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$	$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \nu_i$ $\dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0$

	Hamilton's Principle	Hamilton's Characteristic
With solutions	$Q_i = \beta_i, \quad P_i = \gamma_i$	$Q_i = \nu_i t + \beta_i, \quad P_i = \gamma_i$
Generating Function	$S(q,P,t)$	$W(q,P)$
Hamilton-Jacobi equation	$H\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0$	$H\left(q_i, \frac{\partial W}{\partial q_i}\right) - \alpha_1 = 0$
New constant momenta (one choice)	$P_i = \gamma_i(\alpha_1,...,\alpha_n)$ $\gamma_i = \alpha_i$	$P_i = \gamma_i(\alpha_1,...,\alpha_n)$ $\gamma_i = \alpha_i$
Hamilton-Jacobi solution	$S = S(q_i,\gamma_i,t)$	$W = W(q_i,\gamma_i)$
First half of transformation	$p_i = \frac{\partial S}{\partial q_i}$	$p_i = \frac{\partial W}{\partial q_i}$
Second half of transformation	$Q_i = \frac{\partial S}{\partial \gamma_i} = \beta_i$	$Q_i = \frac{\partial W}{\partial \gamma_i} = \nu_i(\gamma_j)t + \beta_i$

Action-angle Variables

$$J_i = \oint p_i dq_i = \oint \frac{\partial W_i(q_i,\alpha_1,...,\alpha_n)}{\partial q_i} dq_i \qquad w_i = \frac{\partial W}{\partial J_i}$$

Euler-Lagrange equation for fields

$$\frac{\partial \mathfrak{L}}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{\partial \dot{\eta}}\right) - \frac{d}{dx}\left(\frac{\partial \mathfrak{L}}{\partial(\partial \eta/\partial x)}\right) = 0$$

Mathematical identities

$$\sin^2 Q + \cos^2 Q = 1 \qquad \tan^2 Q + 1 = \sec^2 Q$$

$$1 + \cos 2\phi = 2\cos^2 \phi \qquad 1 - \cos 2\phi = 2\sin^2 \phi$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\cot x = \operatorname{cosec}^2 x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

QUESTION 1. (20 marks)

A particle of mass m slides without friction on the inside of a hollow spherical shell of radius a under the influence of gravity.

(a) Use the method of Lagrange multipliers to show that the equations of motion for the system in spherical polar coordinates are given by

$$r: \quad \frac{d}{dt}(m\dot{r}) - \left(mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - mg\cos\theta\right) = \lambda$$

$$\theta: \quad \frac{d}{dt}\left(mr^2\dot{\theta}\right) - \left(mr^2\sin\theta\cos\theta\,\dot{\phi}^2 + mgr\sin\theta\right) = 0$$

$$\phi: \quad \frac{d}{dt}\left(mr^2\sin^2\theta\,\dot{\phi}\right) = 0$$

(b) What are the constraints and constants of the motion?

(c) If the particle starts at the bottom of the sphere with initial velocity v (at $t = 0$, $\theta = \pi$ and $\dot{\theta} = v/a$), identify the velocity range that corresponds to three different types of motion of the particle.

(d) If the particle falls off the interior of the shell, find the value of θ at which this happens for a given initial velocity v.

QUESTION 2. (20 marks)

a) Is the following transformation canonical

$$Q = \frac{1}{q} \qquad P = q^2 p \ ?$$

If not can it be made canonical?

b) Find the $F_2(q,P)$ generating function that generates this canonical transformation.

c) Find a canonical transformation that transforms the Hamiltonian

$$H = \frac{1}{2}\left(\frac{1}{q^2} + p^2 q^4\right),$$

into a new Hamiltonian $K = \frac{1}{2}\left(Q^2 + P^2\right)$. Find the equations of motion for Q and P.

d) If the time evolution of Q is given by $Q(t) = A\sin t$, find the time evolution of q and p in the Hamiltonian in part c) above.

QUESTION 3. (20 marks)

a) The Hamiltonian for a nonlinear system is given by

$$H = \tfrac{1}{2} p^2 + \tfrac{1}{4} \beta q^4 .$$

Find the Hamilton-Jacobi equation for this system.

b) Determine the limits of the periodic motion and show that the action is given by

$$J = 4\alpha^{3/4} \beta^{-1/4} \int_{-1}^{1} \sqrt{1-u^4}\, du .$$

c) What does this result tell us about the action?

d) Use action-angle variables to find the frequency of the motion.

e) If the Lagrangian density for displacements of an elastic rod is given by

$$L = \tfrac{1}{2}\left(\mu \dot{\eta}^2 - Y\left(\frac{\partial \eta}{\partial x}\right)^2 \right)$$

find the Lagrangian equations of motion. What is the physical meaning of this result?

QUESTION 4. (20 marks)

a) Determine the stability of the fixed point at (0,0) for the equations

$$\dot{x} = -y + x\left(x^2 + y^2\right). \qquad \dot{y} = x + y\left(x^2 + y^2\right)$$

b) Change variables to consider the time derivative of r^2. What does this say about the nonlinear stability of (0,0)?

c) The stability of an iterative mapping $x_{n+1} = f(x_n)$ can be determined by calculating the Lyapunov exponent defined by

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln\left|f'(x_i)\right|$$

Find the Lyapunov exponent as a function of μ for the two fixed points of the quadratic map, $x_{n+1} = \mu x_n (1 - x_n)$. Discuss the stability of the fixed points as $\mu \to 2$.

d) What are tangent bifurcations and pitchfork bifurcations and how do they arise?

e) Define the unstable manifold of a fixed point. Explain why unstable manifolds from different fixed points do not intersect.

f) For the following equations

$$\dot{x} = ax + c(y - x)$$
$$\dot{y} = ay + c(x - y)$$

describe the motion when $a > 0$ and $c = 0$.

g) Determine the time evolution of the combination $u = x - y$.

h) What is the physical interpretation of the motion of u when $a < 2c$?